F. R. Bamforth, Grace M. Bareis, I. A. Barnett, P. E. Baur, H. A. Bender, L. T. Black, M. G. Boyce, C. E. Buell, C. T. Bumer, R. S. Burington, W. D. Cairns, E. H. Clarke, Rufus Crane, O. L. Dustheimer, T. M. Focke, B. C. Glover, Harris Hancock, F. C. Jonah, L. C. Knight, H. W. Kuhn, R. H. MacCullough, Florentina Mathias, G. M. Merriman, C. C. Morris, J. R. Musselman, R. L. Newlin, Rufus Oldenburger, Jesse Pierce, H. S. Pollard, C. E. Rhodes, Hortense Rickard, S. A. Rowland, H. E. Stelson, C. F. Thomas, H. A. Toops, M. O. Tripp, J. H. Weaver, R. B. Wildermuth, C. O. Williamson.

The following officers were elected for the coming year: Chairman, O. L. Dustheimer; Secretary-Treasurer, Rufus Crane; Member of Executive Committee, J. H. Weaver; Member of Program Committee, C. O. Williamson.

It is expected that the next meeting of the Section will be held on Thursday, April 6, 1933, at the Ohio State University.

At the afternoon session the following seven papers were presented:

1. "An undergraduate course leading to the study of wave mechanics" by the Chairman of the Section, Professor W. D. Cairns, Oberlin College.

2. "An application of elliptic integrals to a problem in geometry" by C. E. Rhodes, University of Cincinnati.

3. "Canonical multilinear forms with rank as the only invariant" by Rufus Oldenburger, Case School of Applied Science.

4. "A proof of the rule for evaluating an indeterminate form" by C. L. Weaver, Kent State College, by invitation.

5. "Equilateral triangles and squares" by Professor J. R. Musselman, Western Reserve University.

6. "Psychological notions expressed in mathematical form" by Professor H. A. Toops, Ohio State University, department of psychology.

7. "The trihedron and its genesis" by Dr. E. S. Loomis, Professor emeritus, Baldwin Wallace College, by invitation.

Abstracts of these papers follow:

1. Professor Cairns described a two-hour year seminar, conducted this year by the departments of mathematics and physics at Oberlin College, and adapted to senior and first-year graduate students and, as well, selected students who had had six semester hours in calculus and four semester hours in advanced analytic geometry the preceding year. The staff chooses topics in college and first-year graduate mathematics and physics, developing and correlating these with the purpose of giving the students those topics which are needful in beginning the study of wave mechanics. An outline of the topics, with illustrations of assigned problems and drawings (e.g., wave trains), was presented. The course makes it possible for undergraduate students to be ready for an intensive graduate course in wave mechanics.

2. In this paper Mr. Rhodes obtains the geometric interpretation of Landen's transformation given on page 74 of Hancock's Elliptic Integrals from a simple comparison of different methods for finding the volume between a cylinder and a cone.

3. The algebraic theorem on the equivalence of bilinear forms $a_{\alpha\beta}x_\alpha y_\beta$ of rank r to a form $x'_\alpha y'_\alpha$ $(\alpha=1, 2, \cdots, r)$ is generalized in this paper to a theorem on the equivalence of multilinear forms under linear non-singular transformations with coefficients in the field of these forms to canonical types. Similarly to the bilinear case, the matrices of these canonical forms are either identity matrices for certain operations on these forms, or identity matrices bordered by zeros. They are completely characterized by a generalized rank set which insures a factorization property for the associated forms.

4. An indeterminate form expressed as the quotient of two power series can be reduced by synthetic division and the use of the remainder theorem to the derivative form.

5. Professor Musselman discussed equilateral triangles connected with two or three given equilateral triangles or squares. In particular, he proved the two following theorems believed to be new: (a) If $A_1B_1C_1$ and $A_2B_2C_2$ are two directly similar triangles in the plane, if we construct triangles $A_1A_2A_3$, $B_1B_2B_3$, $C_1C_2C_3$ directly similar to the two given triangles, then $A_3B_3C_3$ itself is similar to the others; (b) If $A_1A_2A_3A_4$, $B_1B_2B_3B_4$, $C_1C_2C_3C_4$ are three positively ordered pseudo-squares, i.e., quadrilaterals whose diagonals are perpendicular and equal, if we construct the positively ordered pseudo-squares $A_1B_1C_1D_1$, $A_2B_2C_2D_2$, $A_3B_3C_3D_3$, $A_4B_4C_4D_4$, then $D_1D_2D_3D_4$ is itself a pseudo-square.

6. Professor Toops believes that the teaching of mathematics generally, and of the applied branches especially, is particularly susceptible of improvement through the employment of such tested techniques as: (a) Student practice in "insight," such as seeing the possible permutations of binomials or trinomials in $a+B+C_1+\lambda/\pi$; in analyzing problems without solving them; and in seeing functions plotted in the third dimension as in stereoscopic presentation:[1] (b) Inducing faith in the inevitability and correctness of the end result, and simultaneously making a frontal attack on the initial mental blocking that sometimes affects a student confronted with a new problem, through the several expedients of solving the same problem in many ways in preference to solving many problems each in only one way; of writing out, in the initial attack on a problem, the entire series of steps or operations to be followed, divorced from the concrete numerical application to the problem in question, as a job-analyzed skeleton formula; of estimating numerical results in advance of solving the problem; of employing checking formulae in routine statistical computations, operator checks in tabular substitutions, and degeneracy checks in derivations: (c) Freeing the student from the sheer drudgery of computation through maximal use of calculating machines and computational aids, to the end that the time set free and the time gained through lack of necessity for recomputation because of the accuracy thereby secured, will be such as to enable the student to spend a maximal amount of time in an effort to generalize his results, e.g., as by plotting whole families of nomographic curves while varying the constants systemati-

[1] Vuibert, H., *Les Anaglyphes Géométriques*, Paris, Libraire Vuibert, 1912, 32 pp.

cally, or by rectifying and plotting such rectified systems on log paper or on specially devised coordinates: (d) Intense motivated drill (e.g., by paired comparisons, or permutations of terms, device) aiming at generalizing a specific operation independent of letters and orders, say the squaring of any binomial, as a series of indicated operations, and at freeing the mind to the maximal extent of all rules and crutches by establishing the "verbal set" instead. With the better insight, better background and more time thereby secured, the student also may be encouraged to coin and solve many problems more difficult than those presented in the text or assigned by the instructor: (e) Specific drill in generalization and in application. General transfer of mathematics probably takes place with such great difficulty that the attempt to transfer mathematical thinking specifically is the only remaining hope of efficacious learning, remembering and utilization of mathematics on the part of the average student. Nevertheless, the inductive psychological approach herein advocated will surely improve the attainments of even the sub-average student.

7. Dr. Loomis related the circumstances under which he conceived of an interesting geometric figure, the properties of which he described. Let BAC be the plane angle of a dihedral angle, where $AB = AC$; through B and C draw lines parallel to the edge of the dihedral angle and on these lines take points equally spaced and opposite to each other; these form with A a series of isosceles triangles. Now place each triangle so that its plane is perpendicular to the edge of the dihedral angle and contains the pair of points which determined it, and so that it is symmetrical with the bisecting plane of the dihedral angle. The solid thus formed is bounded by three faces, one generated by the base of the isosceles triangle, being a portion of a hyperbolic cylinder, and two generated by its legs, being conoidal surfaces. It has two planes of symmetry and is infinite in extent in two directions. The consideration of the properties of this figure led Dr. Loomis into some of the properties of space as considered in non-euclidean geometry. It is believed that this figure, which Dr. Loomis calls a "trihedron," does not appear anywhere in the literature of mathematics.

The evening session was given to a study of the work in mathematics that is being done in our secondary schools, as evidenced by the preparation of students entering our colleges. Tests were given on the first day of the fall semester or quarter in September, 1931, in eleven of the colleges and universities of the state to freshman students registered for mathematics. Students to the number of 1446 took this test in this manner, and 642 of these were unable to secure a 50% mark on it. The Section plans to continue these tests and extend the scope of the study, and to make public the results, in an effort to improve the situation.

RUFUS CRANE, *Secretary*

THE ANNUAL MEETING OF THE NEBRASKA SECTION

The annual meeting of the Nebraska Section was held in conjunction with the annual session of the Nebraska Academy of Sciences at the Fontenelle Hotel in Omaha, Friday afternoon May 6th, 1932, with Professor A. K. Bettinger of Creighton University as chairman.

The attendance was about thirty with several students and instructors of both Creighton University and the University of Omaha as guests. Members of the Association present were: A. K. Bettinger, C. C. Camp, A. L. Candy, J. M. Earl, A. L. Hill, J. M. Howie, T. A. Pierce, B. C. Zimmerman.

At the close of the program the following officers were elected for the coming year: Chairman, T. A. Pierce, University of Nebraska; Secretary, A. L. Hill, Peru State Teachers College; Treasurer, J. M. Howie, Nebraska Wesleyan University.

The following papers and reports were presented:

1. "Formulae and algorisms in arithmetic" by B. C. Zimmerman, Creighton University.

2. "A class of non-linear approximating functions" by Professor J. M. Earl, University of Omaha.

3. "Magic squares" by Professor A. L. Candy, University of Nebraska.

4. "Numerical evaluation of infinite series and products" by Professor C. C. Camp, University of Nebraska.

5. "A note on n-ic residues" by Professor T. A. Pierce, University of Nebraska.

Abstracts of some of these papers follow:

1. Mr. Zimmerman presented a variety of algorisms or arrangements in arithmetic operations, showing the great advantage of a left to right procedure in these operations, both in the use of computation formulae, and in the determination of the position of the decimal point in a product or in a quotient.

2. This paper considers the implications of linearity in certain convergence proofs to a class of non-linear functions of approximation.

3. Professor Candy's problem was to construct a magic square of order $2n$ from a given square of order n by what is known as the "Method of current groups." In order to make the process—which is very simple—more conspicuous, heavy lines are used to form the cells of the given square, and the small numbers written at the center of these large cells form the given magic square of order n. Now divide each of these large cells into four small cells thus giving $4n$ cells for the required square of order $2n$. Then in the four cells thus formed within the large cell that contains the number 1, write the first four consecutive numbers 1, 2, 3, 4, and in the four cells within the large cell containing the number 2 write the second four consecutive numbers 5, 6, 7, 8, in large cell No. 3 write the third consecutive group 9, 10, 11, 12. Proceed in like manner until all the cells are filled. These "current groups" are not necessarily "consecutive integers." They may be consecutive terms of any arithmetic series.

But these groups can not be written in these 2×2 squares in any order at random. Here lies the only difficulty in the process. Particular care must be given to the order in which these groups are written, especially if the desired square is to be perfect. There is, however, no necessary order for any group or combination of groups. For this reason it is possible to construct many squares from any given square and with any given set of groups. The orders to be used so as to balance the rows and columns of the required square cannot be made clear in a brief statement.

4. The object of Professor Camp's paper was to present a simple method for finding the numerical value of a series for practical purposes with a minimum of arithmetic. The geometric basis of the inequality given in a former paper[1] was discussed together with further results in the determination of the accuracy by higher analysis of the deviation ratio. For any $p>1$ it was found that the maximum error for $n=10$ in $\sum n^{-p}$ is $<2.3\times 10^{-6}$ and is asymptotic to $p(3-p)/60(n+1)^{p+3}$. By this for $p=2$ the previous error of 2.070×10^{-7} was checked; for $p=3$ when 9 decimals were used an accidental error of 1×10^{-9} was found; and for $p=4$ with 12 decimals and k_{10} to 7 the result was true to 10 decimals. The product $\prod(1-1/n^2)$ was evaluated with 7-place logarithms, k_{10} to 6, and the deviation ratio for the smallest value of p in $\log(1-1/n^2)$, namely $p=2$; and the result 3.676081, obtained instead of the true value, $3.676078=\sinh \pi/\pi$.

5. It is proved in this note that all n-ic residues of an odd prime p are given by $g^{\delta}, g^{2\delta}, \cdots, g^{p-1} \pmod p$ where δ in the greatest common divisor of n and $p-1$ and g is a primitive root of p. This result is used to prove that all n-ic residues of p form a cyclic group.

A. L. HILL, *Secretary*

THE ANNUAL MEETING OF THE MINNESOTA SECTION

The annual meeting of the Minnesota Section of the Mathematical Association of America was held at the State Teachers College, River Falls, Wis., on Saturday, May 7, 1932. Sessions were held at 11:00 o'clock and at 2:15 o'clock with a luncheon at 12:30 o'clock.

Professor Margaret C. Eide, chairman of the Section, presided, except when relieved by Professor Dunham Jackson and Professor R. W. Brink. Ninety persons attended the meeting including the following twenty-four members of the Association: R. W. Brink, W. H. Bussey, Elizabeth Carlson, J. O. Chellevold, H. H. Dalaker, Margaret C. Eide, Grace Erwin, Gladys Gibbens, C. H. Gingrich, W. L. Hart, H. E. Hartig, Dunham Jackson, C. M. Jensen, W. H. Kirchner, Marie M. Ness, A. L. O'Toole, Ole Schey, R. R. Shumway, H. I. Tanjerd, F. J. Taylor, Ella Thorp, A. L. Underhill, Marion B. White, Marian A. Wilder.

At the afternoon session a vote of thanks was adopted as a sign of appreciation of the cordial hospitality of the River Falls Teachers College, and the

[1] Bulletin of the American Mathematical Society, vol. 38 (1932) abstract no. 77, p. 181.

efforts of its department of mathematics. Officers for the following year were elected as follows: Chairman, F. J. Taylor, College of St. Thomas; Secretary, A. L. Underhill, University of Minnesota; Members of the Executive Committee; W. H. Kirchner, University of Minnesota; C. H. Gingrich, Carleton College; Sister Claudette, College of St. Benedict.

The following ten papers were presented:

1. "The operator $\{a^n(x)d^n/dx^n\}^m$" by Dr. M. G. Scherberg, University of Minnesota, by invitation.

2. "On certain singular points of rational curves" by Dr. A. L. O'Toole, National Research Fellow.

3. "Tabulating rational roots and testing possible roots in numerical equations" by Professor F. J. Taylor, College of St. Thomas.

4. "The transformed Fourier equation" by Edwin Oberg, University of Minnesota, by invitation.

5. "The total eclipse of August 31, 1932" by Professor C. H. Gingrich, Carleton College.

6. "Two-dimensional correlation" by Professor Dunham Jackson, University of Minnesota.

7. "Elective courses in applied mathematics for junior and senior engineering students" by Professor M. A. Sadowsky, University of Minnesota, by invitation.

8. "College algebra and trigonometry in the high school" by Professor C. N. Stokes, University of Minnesota, by invitation.

9. "A practical application of complex hyperbolic functions" by Professor H. E. Hartig, University of Minnesota.

10. "Some differential operators in mathematical statistics" by Dr. A. L. O'Toole, National Research Fellow.

Abstracts of these papers follow:

1. The operator $\{a^n(x)d^n/dx^n\}^m$ in which n and m are positive integers, may be written in the form

$$[a_0(a_0+a_1)\cdots(a_0+a_1+a_2+\cdots+a_{m-1})]^n$$

$$=\sum\binom{n}{\alpha_{m-1}}\binom{2n-\alpha_{m-1}}{\alpha_{m-2}}\cdots\binom{in-\alpha_{m-1}-\alpha_{m-2}-\cdots-\alpha_{m-i+1}}{\alpha_{m-i}}$$

$$\cdots\binom{mn-\alpha_{m-1}-\cdots-\alpha_1}{\alpha_0}a_{m-1}^{\alpha_{m-1}}a_{m-2}^{\alpha_{m-2}}\cdots a_1^{\alpha_1}a_0^{\alpha_0}$$

provided $a^n(x)$, an arbitrary function of x, has higher derivatives of order up to and including $n(m-1)$ and the summation is extended over all positive integral and zero values of α_i which satisfy the equation $\alpha_0+\alpha_1+,\cdots,\alpha_{m-1}=nm$ and the inequalities in $-\alpha_{m-1}-\alpha_{m-2}-,\cdots,-\alpha_{m-i+1}\geqq 0$, $i=2, 3,\cdots, m$. The quantities a_i^k are interpreted

$$\begin{cases} a_i^k = a^{k-n}(x)\dfrac{d^k\{a^n(x)\}}{dx^k}, & i,\ k \neq 0 \\ a_0^k = a^k(x)\dfrac{d^k}{dx^k} \\ a_i^0 = 1 \end{cases}$$

A proof of the above theory by mathematical induction was given.

2. This paper consisted in some introductory remarks on a problem of Enriques with some extensions and illustrations. In particular it discussed the conditions under which the rational curves admit a parametric representation by means of polynomials.

3. Professor Taylor commented upon certain methods in the presentation of work in the theory of equations chapter in a college algebra course.

4. If sufficient hypotheses be assumed for $F[x, v(x)]$, then the solutions of the differential equation $d^2u/dx^2+F[x, v(x)]u=0$ can be shown to converge asymptotically to the solution of $\partial^2u/\partial x^2+F(x, 0)u=0$ as x becomes negatively infinite. On applying this theorem to the well-known Bessel and Fourier transformed equations, $\partial^2u/\partial x^2+[e^{2x}-n^2]u=0$ and $\partial^2u/\partial x^2+e^{2x}u=0$ respectively, a straightforward and simple method can be found for finding the solutions of the above equations, whether n be an integer or not.

5. This paper was concerned with the time, the path of totality, and the duration of the eclipse. The features which are of essential interest to the casual observer at the time of a total eclipse of the sun were referred to. The problems which engage the attention of the professional astronomer were briefly reviewed. The fact was mentioned that, after this eclipse, there would be only four more during this century which are total within the United States.

6. This paper is concerned with the coefficient of correlation between the $2n$ coordinates of a set of n points in a plane and the $2n$ coordinates of another such set of n points. If each set is supposed to have its center of gravity at the origin, the coefficient is $+1$ if the configurations are similar and oriented alike, -1 if they are similar but oriented oppositely, and otherwise intermediate between $+1$ and -1. In particular, if one configuration is obtained from the other by rotation through an angle α, the coefficient of correlation is $\cos\alpha$. The analysis is naturally capable of generalization in various ways.

7. The student of engineering spends about two years studying mathematics. College algebra, trigonometry and analytic geometry are taught during the freshman year, differential and integral calculus during the sophomore year. If we analyze the contents of those courses as they are represented by some well known and good text-books, we can see that a great amount of knowledge is accumulated. A junior or senior engineering student will be inclined to go further in theoretical mathematical studies and will be interested to see some applications of the acquired science to engineering. It is obvious, therefore, that elective courses in applied mathematics will be welcomed by juniors and seniors.

This is confirmed by the practical experience which the writer has had in the University of Minnesota. The courses, accompanied by actual exercises in a mathematical laboratory, have been willingly attended by students and have been recommended to them also by some other departments. It is hardly possible to nominate the abundance of topics fitted for presentation in such courses. A few of them are, for instance: Graphical methods for differentiation and integration; graphical treatment of differential equations; computation with approximate values; interpolation and approximation of functions; computation of definite integrals and surfaces by approximate methods; trigonometric series and harmonical analysis; nomography; mathematical instruments. A great many interesting and important topics might be still added to the few just mentioned.

8. Mathematical instruction in the high school seeks to insure facility and accuracy in computation, to consider quantitative aspects of the environment in such a manner that background and meanings are created, and to develop dispositions of and powers in accurate and precise quantitative thinking. It is a platitude to say that the process must be gradual and that the materials through which the aims may be achieved must be presented at the time when they can be learned most economically. For several years both college algebra and trigonometry, per se, have been offered in the twelfth grade of the University of Minnesota High School. The students in these classes have been of the superior type in intelligence. At the completion of the algebra course each class was given the same final examination as that administered in the Arts College of the University. The students in the high school classes occupied, without exception, the tail end of the distribution of scores. Basing the marks upon the classifications used in the college, 3.2 per cent would have received C's, 20 per cent D's, and 76.8 per cent F's. H. W. Bailey,[1] University of Illinois, reports unfavorable results for trigonometry. Some schools, however, have reported favorable data. In some of these instances the replies to inquiries seem to justify the statement that it is one thing to have students learn to be manipulators of symbols and it is an entirely different thing to require them to think in terms of these symbols. Mathematics is a mode of thought. The available data seem to be insufficient to warrant the offering of college algebra and trigonometry in the high school. The high school should do thoroughly what it attempts to do. It should scrutinize its procedures and direct its efforts accordingly. It takes maturity to learn college mathematics, and Bailey contends that repetition means wastefulness as well as poor pedagogy.

9. The problem of determining the voltage and current at every point of a uniform electric line connected to a single frequency sine voltage generator was sketched. The mathematical formulation of the problem and its solution in terms of complex hyperbolic functions were stated. Charts of the complex func-

[1] H. W. Bailey, *Trigonometry in the High School.* The Mathematics Teacher, vol. 25 (1932), pp. 303–308.

tions were shown and their suitability for the solution of the given problem was pointed out.

10. In the development of the mathematical theory of sampling one comes in contact with the tedious work of expressing more or less complicated symmetric functions of the variates in terms of the power sums. And what is worse, one not only must deal with symmetric functions of the variates but with symmetric functions of symmetric functions of the variates. Sets of differential operators which will do the work with a minimum of labour and without the necessity of knowing the expressions for the symmetric functions of lower weight are offered in this paper. See papers by the present author in the Annals of Mathematical Statistics, vol. 2 (1931), pp. 102–149; same journal, vol. 3 (1932), pp. 56–63.

A. L. UNDERHILL, *Secretary*

QUASI-CYCLOTOMIC POLYNOMIALS

D. H. LEHMER,[1] Stanford University

In this paper we consider polynomials of the form

$$f(x) = x^n - a_1x^{n-1} + - \cdots + (-1)^n a_n, \tag{1}$$

where the a's are integers. The polynomial $f(x)$ is said to be quasi-cyclotomic if all its roots lie on a circle[2] of radius r in the complex plane and if the arguments of these roots are commensurable with 2π. In other words each root is r times a root of unity. In case $r=1$, we have the familiar cyclotomic polynomials. Kronecker[3] has shown that cyclotomic polynomials are the only polynomials of the form (1) whose roots lie on the unit circle. For $r \neq 1$, quasi-cyclotomic polynomials are not the only polynomials whose roots lie on a circle of radius r. Further restrictions on the coefficients of $f(x)$ are found to be necessary and sufficient to render $f(x)$ quasi-cyclotomic when its roots lie on a circle.

We shall first give a few theorems for the case $r=1$. Kronecker stated his theorem as follows.

THEOREM 1. *If all the roots of f lie on the unit circle, then they are roots of unity.*

From this theorem Kronecker deduced

THEOREM 1′. *If all the roots of f are real and do not exceed 2 in absolute value, then each root is twice the cosine of a rational multiple of* 2π.

Although Kronecker's proof of Theorem 1 leaves nothing to be desired, it is perhaps of interest to give an independent proof of Theorem 1′ and to deduce Theorem 1 from it.

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[1] National Research Fellow.

[2] All circles in this paper have their centers at the origin.

[3] Kronecker: *Crelle's Journal* 53, 173–175, Werke 1, 105–108. Netto: *Vorlesungen über Algebra* 1, 357–358. Pólya and Szegö, *Aufgaben und Lehrsätze*, II, 149, 368.

The extreme simplicity of Kronecker's proof of Theorem 1 adds not a little to the interest of the theorem. Our proof of Theorem 1′ is quite different from Kronecker's proof. We have tried to make it as elementary as possible, as is also the case with the theorems that follow.

PROOF OF THEOREM 1′. Let m be any integer. From the indentity

$$2\cos m\theta = (2\cos\theta)(2\cos(m-1)\theta) - 2\cos(m-2)\theta$$

follows the well known fact that

$$2\cos m\theta = V_m(2\cos\theta)$$

where $V_m(x)$ is a polynomial of the type (1).

Now suppose that Theorem 1′ is false. Then $f(x)$ has a root

$$\rho_1 = 2\cos 2\pi\omega_1$$

where ω_1 is irrational. Let $h(x)$ be the irreducible[1] factor of $f(x)$ for which $h(\rho_1)=0$ and let

$$\rho_\nu = 2\cos 2\pi\omega_\nu \qquad (\nu = 2, 3, \cdots \mu)$$

be the other roots of $h(x)=0$. Finally consider the product

$$I_m = \prod_{\nu=1}^{\mu} 2\cos 2\pi m\omega_\nu = \prod_{\nu=1}^{\mu} V_m(2\cos 2\pi\omega_\nu) = \prod_{\nu=1}^{\mu} V_m(\rho_\nu).$$

This product is a rational integral symmetric function of the roots of $h(x)$ and hence an integer. Also

$$|I_m| \leqq 2^\mu |\cos 2\pi m\omega_1|.$$

Since ω_1 is irrational we may choose m so that the fractional part of $m\omega_1$, i.e.

$$m\omega_1 - [m\omega_1]$$

is arbitrarily close to $1/4$, so close in fact that

$$|\cos 2\pi m\omega_1| < 2^{-\mu}.$$

With such a value of m, $|I_m| < 1$. But I_m is an integer. Hence $I_m = 0$. This means that some factor $V_m(2\cos 2\pi\omega_\nu)$ of I_m must vanish so that $h(x)$ and $V_m(x)$ have a root in common. Since h is irreducible, all its roots belong to V_m. In particular

$$V_m(2\cos 2\pi\omega_1) = 2\cos 2\pi m\omega_1 = 0.$$

But this contradicts the assumption that ω_1 is irrational.

PROOF OF THEOREM 1. Let us designate the roots of f by

$$e^{i\theta_1}, e^{i\theta_2}, \cdots e^{i\theta_n},$$

[1] In this paper "irreducible" refers to the rational field. The coefficients b_ν of $h(x) = x^\mu + b_1 x^{\mu-1} + \cdots + b_\mu$ are integers by Gauss's Lemma (see for instance, Weber, *Lehrbuch der Algebra* vol. 1, p. 27, or Serret, *Cours d'Algèbre Supérieure* (1885), vol. 1, p. 243.

The coefficients of the polynomial

$$f_1(x) = \prod_{\nu=1}^{n}(x - (e^{i\theta_\nu} + e^{-i\theta_\nu}))$$

are symmetric functions of the roots of f and hence are integers. Its roots, $2\cos\theta_\nu$, do not exceed 2 in absolute value. Therefore by Theorem 1', θ_ν are rational multiples of 2π. Hence the roots of f are roots of unity.

THEOREM 2. *Let the roots of $f(x)$ lie on the unit circle. Writing each root $\rho_\nu = e^{2\pi i\omega_\nu}$ we suppose further that*

$$\frac{1}{8} < |\omega_\nu| < \frac{3}{8}. \tag{2}$$

Then $\omega_\nu = \pm 1/6, \pm 1/4$ or $\pm 1/3$ and $f(x)$ has the form

$$f(x) = (x^2 + 1)^a(x^2 + x + 1)^b(x^2 - x + 1)^c.$$

PROOF. Let $h(x)$ be any irreducible factor of $f(x)$. By Theorem 1, the roots of h are roots of unity. Let $\rho = e^{2\pi ik/m}$, where k is prime to m, be a root of $h(x) = 0$. Finally let

$$Q_m(x) = \prod_{\nu}(x - e^{2\pi i\nu/m})$$

(where ν runs over the positive integers $< m$ and prime to m) be the irreducible factor of $x^m - 1$ which has for roots the primitive mth roots of unity. Then, since h and Q_m are irreducible and have the common root ρ, $h(x) = Q_m(x)$. But for $m \geqq 8$, Q_m and hence h has the root $e^{2\pi i/m}$ which contradicts the first inequality (2). The roots of Q_m for $m = 1$ and 2 are real. Therefore m is reduced to 3, 4, 5, 6 or 7. But Q_5 has the root $e^{2\pi i2/5}$ and Q_7 has the root $e^{2\pi i3/7}$ and

$$3/7 > 2/5 > 3/8,$$

so that for $m = 5$ or 7 the second inequality (2) is violated. Hence $m = 3, 4, 6$ are the only possible values, and

$$Q_4 = x^2 + 1, \quad Q_3 = x^2 + x + 1, \quad Q_6 = x^2 - x + 1.$$

As an analogue of Theorem 2 we have

THEOREM 2'. *Let all the roots of $f(x)$ be real and less than $\sqrt{2}$ in absolute value. Then these roots must be 0 or ± 1 so that*

$$f(x) = x^a(x - 1)^b(x + 1)^c$$

PROOF. Let $\rho_\nu = 2\cos 2\pi\omega_\nu$ ($\nu = 1, 2, \cdots, n$) be the roots of $f(x)$. Since

$$-\sqrt{2} < \rho_\nu < \sqrt{2}$$

it follows that

$$\frac{1}{8} < |\omega_\nu| < \frac{3}{8}.$$

Now the polynomial

$$f_1(x) = \prod_{\nu=1}^{n}(x^2 - \rho_\nu x + 1) = \prod_{\nu=1}^{n}(x - e^{2\pi i\omega_\nu})(x - e^{-2\pi i\omega_\nu})$$

has integer coefficients since they are symmetric functions of ρ_ν. Hence the hypothesis of Theorem 2 is satisfied so that $|\omega_\nu| = \frac{1}{6}, \frac{1}{4}$ or $\frac{1}{3}$. That is $\rho_\nu = 0, \pm 1$.

Before proceeding to generalize Theorem 1 to the case where the roots lie on the circle of any radius it is first desirable to consider what this radius could be. To avoid circumlocution it is helpful to make the following

ASSUMPTION. *$f(x)$ is not a polynomial in x^k where $k>1$.* In the opposite case we can simplify matters by replacing x^k by x thus replacing the roots by their kth powers. This transformation throws a set of roots having the same absolute value into another such set without altering the rational or irrational character of the arguments of the roots. Having agreed on this assumption we proceed to prove

THEOREM 3. *Let the roots of f lie on a circle of radius r. Then r is an integer or the square root of an integer according as n is odd or even.*

PROOF. It follows from our assumption that f has a coefficient $a_h \neq 0$ for which h is prime to n. Writing the roots of f in the form $\rho_\nu = re^{i\theta_\nu}$ (where θ_ν may be zero) we consider the symmetric function

$$a_{n-h} = a_n \sum(\rho_1\rho_2 \cdots \rho_h)^{-1} = r^{n-h} \sum e^{-i(\theta_1+\theta_2+\ldots+\theta_h)}$$
$$= r^{n-h} \sum e^{i(\theta_1+\theta_2+\ldots+\theta_h)} = r^{n-2h}a_h.$$

That is

$$r^{2h}a_{n-h} = a_n a_h. \tag{3}$$

Raising both sides to the nth power we get

$$a_n^{2h} a_{n-h}^{n} = a_n^{n} a_h^{n}.$$

Since $a_h \neq 0$, a_n^{2h} is a perfect nth power. If n is odd, $2h$ is prime to n. Hence $a_n = r^n$ is the nth power of an integer. If n is even h is prime to n. Hence $a_n^2 = (r^2)^n$ is the nth power of an integer. From this the theorem follows.

In order to show that r need not be an integer we have only to exhibit the case $f(x) = x^2 + 2x + 2$.

The question now arises: If the roots of f lie on a circle, are their arguments necessarily commensurable with 2π as in the case when the circle is the unit circle? The example $f(x) = x^2 - x + 4$ shows that the answer is no. Our problem is to find a necessary and sufficient condition on f in order that its roots be proportional to roots of unity. We consider first the case in which r is an integer.

THEOREM 4. *If all the roots of $f(x)$ lie on the circle of integer radius r, then in order that these roots be proportional to roots of unity, it is necessary and sufficient that a_ν be divisible by r^ν.*

PROOF. Define b_ν by $a_\nu = b_\nu r^\nu (\nu = 1, 2, \cdots, n)$. Then the polynomial whose roots are $1/r$ times the roots of $f(x)$ is

$$g(x) = f(rx)/r^n = x^n - b_1 x^{n-1} + \cdots + (-1)^n b_n$$

and has all its roots on the unit circle. If the b's are integers these roots are roots of unity by Theorem 1. Conversely if the roots are roots of unity, the b's must be integers since roots of unity satisfy irreducible equations with integer coefficients. This proves the theorem.

In case r is not an integer, the situation is slightly more complicated. It is convenient to write $r = s\sqrt{R}$, where s and R are integers and R has no square factor > 1. We have then the following counterpart of Theorem 4.

THEOREM 5. *Let all the roots of f lie on a circle of radius $s\sqrt{R}$. Then in order that these roots be proportional to roots of unity it is necessary and sufficient that a_ν be divisible by*[1] $s^\nu R^{[(\nu+1)/2]}$.

PROOF OF NECESSITY. Let the roots of f be $s\sqrt{R}\,\epsilon_\nu$ where ϵ_ν are roots of unity. We consider the polynomial

$$g(x) = f(sx)/s^n = x^n - \frac{a_1}{s} x^{n-1} + \cdots + (-1)^n \frac{a_n}{s^n}$$

whose roots are $\sqrt{R}\,\epsilon_\nu$. Our first task is to show that the coefficients of $g(x)$ are integers. Let $a_\nu/s^\nu = b_\nu$ be its νth coefficient. Consider next the polynomial $g_2(x)$ whose roots are the squares of the roots of $g(x)$, so that $g_2(x^2) = g(x)g(-x)$.

$$g_2(x) = \prod_{\nu=1}^{n} (x - R\epsilon_\nu^2) = \sum_{\nu=0}^{n} (-1)^\nu x^{n-\nu} A_\nu$$

where[2]

$$A_\nu = (-1)^\nu b_\nu^2 + 2\sum_{i=0}^{n} (-1)^i b_{2\nu-i} b_i \quad (\nu > 0), \quad A_0 = 1. \tag{4}$$

Hence A_ν are rational numbers. Finally the coefficients of the polynomial $g_3(x) = g_2(xR)/R^n$ are rational while its roots, ϵ_ν^2, are roots of unity. Hence its coefficients as well as those of $g_2(x)$ must be integers. Thus $g(x)g(-x) = g_2(x^2)$ has integer coefficients. Hence by Gauss' lemma the coefficients b_ν of $g(x)$ are integers. To complete the proof it is sufficient to show that b_ν is divisible by $R^{[(\nu+1)/2]}$.

By Theorem 3, n is even. Let $n = 2k$. Also (3) can be written

$$a_{n-\nu} = R^{k-\nu} a_\nu. \tag{5}$$

Now by Theorem 4

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[1] Here $[x]$ denotes, as usual, the greatest integer $\leqq x$.

[2] In (4) $b_j = 0$ for $j < 0$ and $j > n$; $b_0 = 1$.

$$A_\nu = R^\nu B_\nu. \tag{6}$$

If in (4) we set $\nu = k$ and make use of (5) and (6) we get

$$A_k = R^k B_k = (-1)^k a_k^2 + 2\sum_{i=0}^{k-1}(-1)^i a_i^2 R^{k-i}. \tag{7}$$

Hence R divides a_k^2. But R has no square factor. Therefore R divides a_k. Writing $\nu = k-1$ in (4) we have

$$A_{k-1} = R^{k-1}B_{k-1} = (-1)^{k-1}a_{k-1}^2 + 2\sum_{i=0}^{k-2}(-1)^i a_i a_{i+2} R^{k-i-2}.$$

Since R divides a_k, R divides the summation. Hence a_{k-1} is divisible by R. Similarily we may show that R divides $a_{k-2}, a_{k-3}, \cdots, a_1$. Fortified by this knowledge we return to (7). The summation is now divisible by R^3 and hence by R^4. Hence R^2 divides a_k. As before $a_{k-1}, a_{k-2}, \cdots, a_3$ are divisible by R^2. We now return again to (7) and repeat the reasoning until we find that for $\nu \leqq k$, a_ν is divisible by $R^{[(\nu+1)/2]}$. Finally we use (5) and find that $a_{n-\nu}$ is divisible by R^λ, where

$$\lambda = [(\nu+1)/2] + k - \nu = [(n-\nu+1)/2].$$

Hence our condition is necessary.

PROOF OF SUFFICIENCY. Let a_ν be divisible by $s^\nu R^{[(\nu+1)/2]}$. Then b_ν is divisible by $R^{[(\nu+1)/2]}$. Hence from (4), we see that A_ν contains R^μ where

$$\mu = \min_{i \leqq \nu}\left\{\left[\frac{2\nu - i + 1}{2}\right] + \left[\frac{i+1}{2}\right]\right\} = \nu.$$

Theorem 4 then tells us that the roots of $g_2(x)$ and hence those of $f(x)$ are proportional to roots of unity. This proves the sufficiency of our condition.

The totality of all polynomials of degree n with integer coefficients whose roots are roots of unity multiplied by the square root of an integer, is, of course, infinite, since we may multiply the roots by any integer. A polynomial f whose roots lie on a circle of radius $r = \sqrt{R}$, where R has no square factor may be thought of as the representative of the class of all those polynomials whose roots are integer multiples of the roots of f. The following theorem shows that the number of such classes is actually finite.

THEOREM 6. *Let the $n = 2k$ roots of $f(x)$ be of the form $\sqrt{R}\epsilon_\nu$ where ϵ_ν are roots of unity, and where R has no square factor. Then*

$$\sqrt{R} \leqq \frac{(2k)!}{(k+1)!(k-1)!}.$$

PROOF. Since f is not a polynomial in x^2 or x^k it has a coefficient $a_\nu \neq 0$ such that ν is odd and $< k$. Then by Theorem 5

$$a_\nu = b_\nu R^{[(\nu+1)/2]} = \sum R^{\nu/2}\epsilon_1\epsilon_2 \cdots \epsilon_\nu .$$

Hence

$$|a_\nu| = |b_\nu| R^{[(\nu+1)/2]} \leqq R^{\nu/2}\binom{n}{\nu}.$$

That is

$$|b_\nu| \sqrt{R} \leqq \binom{n}{k-1} = \frac{(2k)!}{(k-1)!(k+1)!} .$$

Since $|b_\nu| \geqq 1$, the theorem follows. A rather crude estimate for R is

$$R < \frac{4^n}{\pi(n+4)} .$$

Examples of quasi-cyclotomic polynomials whose roots lie on a circle of radius $\sqrt{R}$ are

$$R = 2\colon x^2 + 2x + 2$$
$$R = 2\colon x^4 + 2x^3 + 2x^2 + 4x + 4$$
$$R = 5\colon x^4 + 5x^3 + 15x^2 + 25x + 25$$
$$R = 6\colon x^4 + 6x^3 + 18x^2 + 36x + 36$$
$$R = 7\colon x^6 + 7x^5 + 21x^4 + 49x^3 + 147x^2 + 343x + 343 .$$

THE INDIAN ORIGIN OF THE MODERN PLACE-VALUE ARITHMETICAL NOTATION. PART II

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It has been shown in previous issues[1] of this MONTHLY that the works written by the elder Āryabhaṭa and some of his successors contain clear evidence of the employment in India of the modern place-value arithmetical notation at least as early as 499 A.D. Further examination of the works confirms this conclusion. Varāhamihira speaks of the addition and subtraction of the zero.[2] Brahmagupta gives the following rules for different operations with the zero:[3]

(a) The sum of a negative quantity and the zero is negative; the sum of a

[1] Vol. 34, p. 409, and vol. 39, p. 251.

[2] *Pañca-siddhāntikā*, III. 17; IV. 7, 8, 11, 12; XVIII. 35, 44, 45, 48, 51.

[3] The reference is to Sudhākar Dvivedī's edition (Benares). In the preface the editor writes that he prepared this edition after consulting three manuscript copies of the work. Colebrooke's edition is based only on one copy found in the India Office Library. The serial numbers of the verses giving these rules in Colebrooke's edition are the numbers immediately following those in Dvivedī's edition.

positive quantity and the zero is positive; and the sum of two zeros is zero (*Brāhma-sphuṭa-siddhānta*, XVIII.30).

(b) On being diminished by the zero, a negative quantity becomes negative, a positive quantity becomes positive, and the zero becomes zero (*Ibid.*, XVIII. 32).

(c) The product of the zero and a negative quantity or of the zero and a positive quantity or of two zeros is zero (*Ibid.*, XVIII.33).

(d) The zero divided by the zero is zero (*Ibid.*, XVIII.34).[1]

(e) Khodhṛtamṛṇaṃ dhanaṃ vā tacchedaṃ khamṛṇadhanavibhaktaṃ vā (*Ibid.*, XVIII.35).

When a positive or negative quantity is divided by the zero, the quotient is *taccheda*[2] (*khaccheda*)—an indeterminate quantity of which the zero is the divisor. When the zero is divided by a positive or negative quantity, the quotient is *taccheda* (*khaccheda*)—a fraction of the zero, *i.e.*, zero.

(f) The square root of zero is zero; the square of zero is the same as its square root (*Ibid.*, XVIII.35).

If a place-value notation with the zero to indicate a vacant place had not existed at the time of Varāhamihira and Brahmagupta, there would have been no necessity for the former to speak of addition and subtraction of the zero and for the latter to give the rules quoted above. Consider the numbers 'five hundred and seven' and 'four hundred and two.' In finding the sum of these two numbers hundreds and units are naturally added separately so that the result becomes 'nine hundred and nine.' The question of addition of zeros can arise only when the numbers are expressed in a place-value decimal notation with the zero indicating a vacant place. Similar is the case with most other arithmetical operations with the zero. The above-mentioned rules, therefore, presuppose the existence of the modern arithmetical notation in India at the time of Varāhamihira and Brahmagupta.

The above conclusion is further supported by the use of the word *aṅka* for the number *nine* in Varāhamihira's *Pañcasiddhāntikā* (XVIII.35), Brahmagupta's *Brāhma-sphuṭa-siddhānta* (II.7) and Lalla's (7th century A.D.) *Śiṣyadhi-vṛddhida*.[3] As there is only one moon or one earth, the word *śaśī* (the moon) or *bhū* (the earth) has been used to denote the number *one*. As a man has two hands and two eyes, the word *kara* (hand) or *nayana* (eye) has been used to denote the number *two*. Similar considerations have restricted the choice of words

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[1] This rule shows that Brahmagupta did not take the zero to mean an infinitesimal quantity. In deducing this rule from the rule $0\times0=0$ he has ignored the rule $0\times a=0$.

[2] The compound word *taccheda* (=*tat*+*cheda*) has two different meanings according to the kind of *samāsa* (compounding of words) used to form it. If the *samāsa bahuvrīhi* is used, it means a quantity of which the zero is the divisor; for, *tat* (*i.e.*, it) refers to the zero. If the *samāsa tatpuruṣa* is used, it means *tasya* (of the zero) *chedaḥ* (a fraction), *i.e.*, a fraction of the zero. In the former sense Brahmagupta's *taccheda* is the same as Bhāskara's *khahara* (infinity).

[3] Chapter I (Graha-gaṇitādhyāya), Section I, Verses 53 and 59; Section II, Verses 2, 5, 8, 9, and 28; Section III, verse 9; etc.

for other numbers. Hence the use of the word *aṅka*, for the number *nine*, points to the fact that the number of *aṅkas* or figures used at the time of these writers must have been nine. Previous Indian notations required many more signs or figures. It is, therefore, concluded that the notation employed by these writers required nine significant figures with the zero and was, therefore, the modern place-value decimal notation. When, therefore, Severus Sebhokt in the year 662 A.D. speaks admiringly of the Hindu method of computation by means of nine signs,[1] he means the employment of the modern place-value decimal notation.

Oriental scholars (whether American, European, or Indian) who had sufficient knowledge of Sanskrit to be able to examine for themselves astrological and astronomical works written by Varāhamihira and his successors were definitely of the opinion that the modern notation was in use in India as early as the sixth century A.D. Even G. R. Kaye—the most uncompromising supporter of the theory of the non-Indian origin of the modern notation—did not doubt the authenticity of the Sanskrit texts of the works of Varāhamihira and Brahmagupta as handed down to us. It will thus be seen that Cajori is not justified in remarking that the manuscript of Brahmagupta's work used by Colebrooke "cannot be accepted as evidence that Brahmagupta himself used the zero and the principle of local value,"[2] unless by 'the manuscript' he means the numbers expressed in the modern notation by means of decimal figures, which occurred in the manuscript by the side of the corresponding number-expressions in terms of word-numerals based on the modern notation.

It may be pointed out here that, in the case of numbers and dates included in verse, word-numerals had to be used instead of decimal figures. But the underlying notation was the modern place-value notation. In arithmetical operations signs are far more advantageous than words. As there is indisputable evidence to show that numeral signs have been in use in India from before the Christian era, there is no reason to suppose that in performing arithmetical operations the Indians from the sixth century onwards did not use the modern notation with decimal figures. It is well-known that there have always existed in India two parallel systems of expressing numbers, arising from the same notation—one by means of words or letters so as to be capable of being used in verse and the other by means of figures so as to be most suitable for calculation. Cajori is quite right when he observes, "There appear to have been several notations in use in different parts of India, which differed, not in principle, but merely in the forms of the signs employed."[3] The genealogical table given below will demonstrate the truth of this observation and show how the different Indian systems of expressing numbers by means of figures, word-numerals, or letter-numerals, have arisen, by successive natural stages, from the ordinary method of stating numbers in words.

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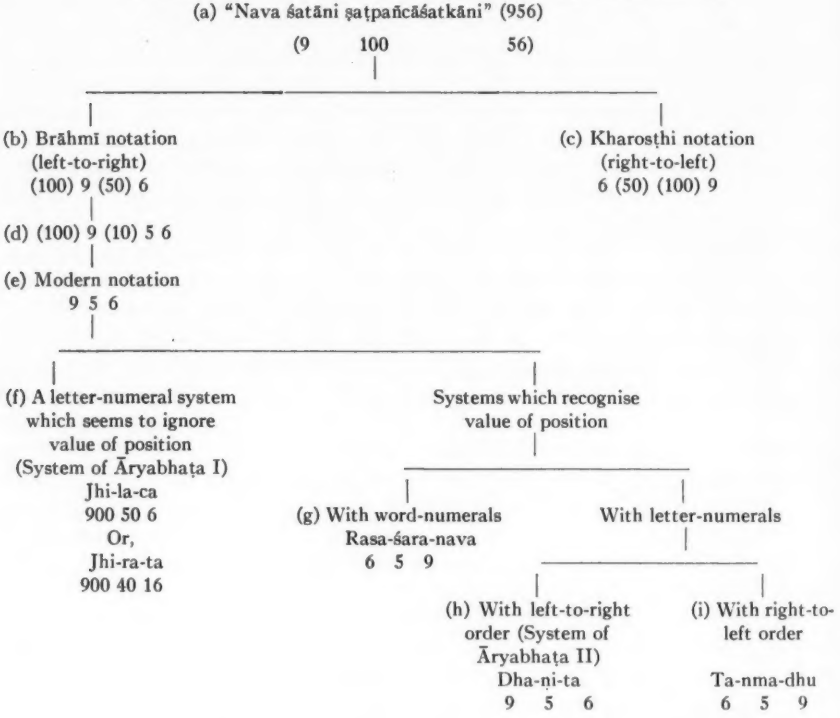
[1] Cajori, *A History of Mathematical Notations*, vol. I (1928), p. 48.
[2] *Ibid.*, Article 76.
[3] *A History of Mathematics* (1922), p. 89.

A genealogical table of different Indian systems of expressing numbers


(a) "Nava śatāni ṣaṭpañcāśatkāni" (956)
(9 100 56)
(b) Brāhmī notation (left-to-right) (100) 9 (50) 6
(c) Kharoṣṭhi notation (right-to-left) 6 (50) (100) 9
(d) (100) 9 (10) 5 6
(e) Modern notation 9 5 6
(f) A letter-numeral system which seems to ignore value of position (System of Āryabhaṭa I) Jhi-la-ca 900 50 6 Or, Jhi-ra-ta 900 40 16
Systems which recognise value of position
(g) With word-numerals Rasa-śara-nava 6 5 9
With letter-numerals
(h) With left-to-right order (System of Āryabhaṭa II) Dha-ṇi-ta 9 5 6
(i) With right-to-left order Ta-nma-dhu 6 5 9

In the above table it should be understood that (100), (50), and (10) mean single symbols for these numbers; (100)9 means a single compound symbol for 900 obtained by prefixing the symbol for 100 to that for 9.

The expression "nava śatāni ṣaṭpañcāśatkāni" for the number 956 has been taken from the Caraka[1] *Saṃhita* (Śārīrasthānam, Ch. VII, 13) where the number 29956 has been expressed in words as "ekonatriṃśat sahasrāṇi nava ca śatāni ṣaṭpañcāśatkāni." The notations (b), (c), (e), (g), and (i), occur in Indian inscriptions. Kaye says that he has not found epigraphical examples of the system (f).[2] I do not know whether the notations (d) and (h) may be found in in-

[1] Caraka was the official physician of King Kaniṣka in the first century A.D. (Macdonell's *History of Sanskrit Literature*, (1917), pp. 435, 436.)

[2] I think that this sytem was never used in any inscription. It is very doubtful whether its author used it except in the *Daśagītikā*. Bhaṭṭotpala (966 A.D.) quotes a verse from the elder Āryabhaṭa in his commentary on Varāhamihira's *Vṛhat-saṃhitā*. The first half of the verse is found in the *Āryabhaṭīya* but not the second half. A later work of Āryabhaṭa has been lost (*Bulletin* of the Calcutta Mathematical Society, Vol. XXII, pp. 115–120). It is very probable that Bhaṭṭot-

scriptions. Notations (f) and (h) have been explained respectively by the elder[1] and the younger Āryabhaṭa.[2] The notation (d) is certainly the connecting link between (b) and (e), and is easily obtained as shown in the above table.[3] No development of the Kharoṣṭhi notation could take place as, according to Marshall,[4] the Kharoṣṭhi script disappeared from India in the fifth century A.D.

At first sight there appears to be no connection between the modern notation (e) and the letter-numeral system (f) of the elder Āryabhaṭa. The case of this system is somewhat similar to that of the arithmetical operation 'multiplication' which "is not necessarily addition, but an operation self-contained, self-interpretable—springing originally out of addition, but, when full-grown, existing apart from its parent."[5] Just as the relation between multiplication and its parent, *viz.*, addition, is apparent only when the multiplier is a positive integer, so the relation between the alphabetic system of the elder Āryabhaṭa and its parent, *viz.*, the modern place-value notation, is noticeable only in expressions like *jhi-la-ca* for 956 and *yu-gu-śi-khi-la-ga* for 337253.

It will appear from the above genealogical table of different Indian notations that the abacus is not an essential pre-requisite to the modern place-value arithmetical notation.

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pala quotes from this lost work. The second half of the verse states the number 364224 in word-numerals as *jina-yama-veda+ṛtu-havyabhuk* (24, 2, 4, 6, 3). If this view be correct, the elder Āryabhaṭa must be given the credit of also first showing the method of constructing number-expressions which occur in the works of subsequent Indian authors.

[1] *The Āryabhaṭīya*, Chapter I (*Daśagītikā*), Verse 2.

[2] *Mahāsiddhānta*, Chapter I, Verse 2.

[3] This suggestion seems to be supported by the Tamil numeral figures found in inscriptions and manuscripts of the 16th century. These figures "are remarkable as forming the stage of development between the W. Cave numerals and the modern systems, and are, therefore, relics of a system that became more or less obsolete in the sixth century A.D.; we find here separate figures for *ten*, *hundred* and *thousand* nearly identical with the W. Cave forms; but the figures for *twenty*, etc. are rejected, and tens, hundreds or thousands are expressed by prefixing the sign for the units to the left side of the figure representing the order." (Burnell, *South Indian Palæography*, p. 68). Also see Humboldt quoted by Cajori in his *History of Mathematical Notations*, Vol. I (1928), Art. 88. In the Brāhmī notation hundreds as well as thousands used to be expressed by affixing the sign for the units to the right side of the figure representing the order. There is no reason why the order in which these symbols occurred in the notation should not have been retained when the symbols were separated.

[4] *Memoirs of the Archæological Survey of India*, No. 7, p. 10.

[5] Kelland and Tait, *Quaternions*, Introduction.

NOTES ON CODE WORDS

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Introduction

The following notes are believed to be of interest in connection with the present regulations governing the construction of code words, as adopted in 1928 by the International Telegraph Conference of Brussels.[1] Briefly stated, two types of code language are permissible under the present regulations. We shall proceed first to a consideration of words of the first type, called *Category A* words, and conclude with a brief discussion of words of the second type, called *Category B* words, which are, it may as well be stated at once, of far less importance than the Category A words.

That part of the protocol adopted by the 1928 Telegraph Conference dealing with the requirements which words of Category A must fulfill reads as follows:[2]

Category A. Telegrams the text of which contains code words formed of a maximum of 10 letters and in which there is at least one vowel when they have a maximum of 5 letters, at least 2 vowels when they have 6, 7, or 8 letters, and at least 3 vowels when they have 9 or 10 letters. In words of more than 5 letters there must be at least one vowel in the first 5 letters, and at least one vowel in the remainder of the word, it being understood that words of 9 or 10 letters must contain at least a total of 3 vowels. The vowels are a, e, i, o, u, y.

A number of queries suggest themselves in connection with words of this category. In order to understand the origin and significance of these queries, a brief history[3] of code language as employed in international telegrams must first be presented.

It is well known that the basis of all or practically all modern cable and telegraph codes is the 5-letter code word. Through a loophole unforseen by those who participated in formulating the rules drawn up by the International Telegraph Conference of London, in 1903, it became possible to combine two 5-letter code words to form a 10-letter word chargeable as a single word in cablegrams, thus cutting the cost of messages in half. Code compilers and code users were very quick to find and take advantage of this loophole, with the result that

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[1] The present regulations went into effect on October 1, 1929. That they are inadequate and that they have failed to accomplish the reforms intended is sufficiently attested to by the fact that proposals for their further modification will constitute one of the most important subjects on the agenda for the forthcoming International Telegraph Conference, which is scheduled to open in September of this year at Madrid.

[2] Code words of this type must conform to other requirements not mentioned herein, but they are of no interest in the present discussion.

[3] For a more detailed history see Friedman, William F., *The history of codes and code language, the international telegraph regulations pertaining thereto, and the bearing of this history on the Cortina report*, Government Printing Office, Washington, 1928.

within a short time 5-letter codes became very widespread and have by this time practically superseded all other types of codes in international cable and radio communications. Thus, while theoretically the 1928 regulations governing Category A code language are applicable to codes based upon words of a length up to 10 letters, in practice they are aimed at codes based upon 5-letter words which, as every code user now knows, are to be combined in pairs to form singly-charged-for 10-letter words. Now let us suppose that in a given code there are some 5-letter words with only one vowel. (With one or two exceptions, this is actually the case in the 5-letter codes constructed up to 1929.) It follows that in a certain number of cases there will be brought together, in the pairing of two 5-letter words to form a single 10-letter word, two words each containing a single vowel, resulting in the formation of a 10-letter word with only two vowels. According to the present regulations, such a word would have to be charged for as two words, thus increasing the cost of messages to an extent more or less dependent upon the number of code words containing only one vowel.[1] The only absolutely sure way of avoiding this source of surcharge is to arrange that *every* word in the code contain at least two vowels. Thus, while the regulations do not specifically state that each 5-letter code word of Category A must contain at least two vowels, it is clear that, in order to insure that the pairing of two 5-letter groups will in *no case* result in the formation of a 10-letter word with only two vowels, *each 5-letter word must contain at least two vowels*. The new codes (those published since October 1, 1929) take this indirectly imposed requirement into account.

Thus far we have been dealing with certain limitations imposed upon codes by the regulations themselves. We come now to the consideration of a very important limitation imposed upon modern codes as a result of practical difficulties inherent in telegraphic communication.

From the earliest days of codes based upon the 5-letter artificial word, it was recognized that some method or device is necessary whereby errors of transmission can be more or less automatically detected and corrected by the recipient of a message; otherwise communication by means of such artificial words becomes impractical. Of the methods elaborated for this purpose, that based upon the principle of including within a given code only such words as differ from one another in at least 2 letters has proved to be the most satisfactory, and has practically superseded all other methods. The 2-letter difference may consist in:

(1) A difference in the identity of 2 letters. For example, if the code contains the word ABABA, it must not contain any word differing from it in only one letter, such as ABABE, ABACA, ABEBA, ACABA, or EBABA. But words

[1] It is incorrect to assume that the number of 10-letter words that will be subject to the double charge is directly dependent upon the actual number of single-vowel, 5-letter words present in the code. Only an actual test made upon many bonafide messages, all prepared with the same code, can satisfactorily determine the number of doubly-charged-for words to be expected for that code, because the words and phrases of any code are employed with greatly varying frequencies.

differing from ABABA in at least two letters, such as ABACE, ABEBE, ACEBA, ECABA, ABECA, etc., are permissible.

(2) A difference in the position of 2 letters. For example, if the code contains the word ABECI, it may contain words such as EBACI, ACEBI, ABICE, BAECI, AEBCI, etc. In the early codes, no attempt was made to eliminate or to suppress one or the other member of a pair of words differing simply in the positions occupied by two adjacent letters. Many of them contained groups, such as ABECI and ABEIC, which are easily converted one into the other by a common type of psychological *lapsus calami* referred to in code work as "transposition." But in the better codes constructed up to about the year 1925, the authors have usually recognized the necessity of avoiding the possibility of errors introduced by a transposition of adjacent letters, and, to a very large extent, have succeeded in eliminating or almost completely suppressing this source of error. Something will be said of this later in these notes. In the more modern codes serious attempts have been made to eliminate errors due to transpositions of both adjacent and alternate letters, and it may be said that the problem is rather a difficult one.

(3) A difference in the identity of one letter and in the position of another. For example, if the code contains the word ABECK, the following would be legitimate words in the same code: ABERC, AERCK, etc.

In short, when at least two homologous letters in a pair of code groups differ in their identities, the code words are said to present a 2-letter difference.

In what follows we shall refer to *classes* and *subclasses* of words. By the designation *class*, we refer to a set of words merely by indicating the *number* of their constituent vowels and consonants. Using the symbol C to represent any of the 20 consonants, and the symbol V to represent any of the 6 vowels, the designation "$3V/2C$" merely indicates that words of this class contain 3 vowels and 2 consonants, without indicating the positions occupied by any of these constituent elements. Under each class, except the first and last, there are different subclasses of words with respect to the exact arrangement or *position* of the vowels and the consonants composing them. In the accompanying Table I there are shown the six classes and the 32 subclasses of words which can be constructed by taking vowels and consonants in groups of five letters.

If we wish to compose 5-letter words imposing no limitations other than that they must show a 1-letter difference, we may place any one of 26 letters in each of the five positions and shall thus have a total of $26\times26\times26\times26\times26=11{,}881{,}376$ words. We may consider that we are here concerned with a simple multiplication involving five factors the numerical value of each of which is equal to the number of elements available for permutation.

If, however, we now impose the limitation that the words shall show a difference of two letters, the number of words that can be composed will be considerably reduced, because, in the parlance of code compilers, it is necessary to "sacrifice one of the letters." Just what this means will now be explained.

TABLE I

Class No.	Class Description	Subclass No.	Subclass Description
I	5V	1	VVVVV
II	4V/1C	2	VVVVC
		3	VVVCV
		4	VVCVV
		5	VCVVV
		6	CVVVV
III	3V/2C	7	VVVCC
		8	VVCVC
		9	VCVVC
		10	CVVVC
		11	VVCCV
		12	VCVCV
		13	CVVCV
		14	VCCVV
		15	CVCVV
		16	CCVVV
IV	2V/3C	17	CCCVV
		18	CCVCV
		19	CVCCV
		20	VCCCV
		21	CCVVC
		22	CVCVC
		23	VCCVC
		24	CVVCC
		25	VCVCC
		26	VVCCC
V	1V/4C	27	VCCCC
		28	CVCCC
		29	CCVCC
		30	CCCVC
		31	CCCCV
VI	5C	32	CCCCC

Consider for example, code words of two letters. Obviously, with an alphabet of 26 letters a total of 26×26, or 676, 1-letter difference pairs can be constructed. Such pairs will represent *all* the permutations of the 26 letters taken 2 at a time, such as AB, BA, AC, CA, etc. But if a 2-letter difference is desired, then only a total of 26 different pairs can be constructed, whether the pairs be simple doublets, such as AA, BB, $\cdots$, ZZ, or permutations of 2 different letters, such as AB, BC, $\cdots$, ZA. Thus, the formula for 1-letter difference pairs, 26^2, becomes modified to $26^{(2-1)}$ in order to take care of a 2-letter difference. Of the two factors in the multiplication 26×26, one will be reduced to unity, which is what the code compiler really means when he says that "one of the letters must be sacrificed." Similarly, in the case of 3-letter words, the total number of 1-letter difference words is 26^3 or 17,576 words; but if a 2-letter difference is desired the total number becomes reduced to $26^{(3-1)}=26^2=676$ words. For 4-letter groups with a 2-letter difference the number becomes 26^3 words; and for 5-letter groups with a 2-letter difference, 26^4 or 456,976 words. A general formula may be derived from the foregoing:

$$\text{Number of 2-letter difference words} = \lambda^{(n-d+1)},$$

where λ = the number of elements in the alphabet, n = the number of characters per word, d = the differential.

So far we have imposed no restrictions on the *kind* of letter that may occupy any given position in a word: any letter may be a vowel or a consonant as we happen to take it. If we restrict the number of vowels or the number of consonants allowed to a word, the case is altered. When we demand a consonant we are limited to 20 letters instead of 26, and when we demand a vowel, we are

limited to 6 letters instead of 26. If we wish to form *all* the words that can be composed containing, say 1 consonant followed by 4 vowels, i.e., words of the form $CVVVV$, we shall have $20\times6\times6\times6\times6=25{,}920$ words each differing from all the others in at least one letter.

If, in addition to limiting ourselves to words consisting exclusively of one consonant followed by four vowels, we make the further demand that our words shall differ among themselves in at least two letters, we shall further lessen their number. We have already seen that the requirement of a 2-letter difference entails the sacrifice of one factor, or rather the reduction of one factor to unity. Just what this sacrifice will mean when we limit ourselves to a certain number of vowels and a certain number of consonants in each word will become plain as we proceed.

Query I

What is the theoretical maximum number of 5-letter code words that can be constructed with a minimum 2-letter difference, each code word containing a minimum of 2 vowels?

Solution

1. Table II shown below sets forth in brief form all the classes of 5-letter code words and the maximum number of 2-letter difference words in each class. Some explanation as to how the totals for each class of words are derived is added.

TABLE II

Based upon an Alphabet of 20 Consonants and 6 vowels

	Description	*Maximum number of 2-letter difference words*
(1)	Class I—With $5V$ $=6\times6\times6\times6\times1=$ 1,296	$=$ 1,296
(2)	Class II—With $4V/1C=6\times6\times6\times6\times1=$ 1,296;	1,296$\times$ 5$=$ 6,480
(3)	Class III—With $3V/2C=6\times6\times6\times20\times1=$ 4,320;	4,320$\times$10$=$ 43,200
(4)	Class IV—With $2V/3C=6\times6\times20\times20\times1=$ 14,400;	14,400$\times$10$=$144,000
(5)	Class V—With $1V/4C=6\times20\times20\times20\times1=$ 48,000;	48,000$\times$ 5$=$240,000
(6)	Class VI—With $5C$ $=20\times20\times20\times20\times1=$160,000	$=$160,000

(1) In Class I (words composed of 5 vowels) the number of words is the same as if we were forming our words from a 6-letter alphabet. If we allow for a 2-letter difference, which means that we must reduce one of the five factors to unity, we shall, accordingly, have $6\times6\times6\times6\times1=1{,}296$ words.

(2) Class II consists of words composed of 1 consonant and 4 vowels. There are 5 subclasses according to the position occupied by the consonant. If we take the consonant as the first letter of each word, we can divide our 5 letters into two groups—one of 2 letters, CV, and one of 3 letters, VVV. If we use all the VVV groups, we shall have $6\times6\times6=216$ VVV groups. Each of these must now be combined with as many CV groups as possible to make 5-letter words. Each

of the *CV* groups with which any one *VVV* group is combined must differ from the other groups combined with the same *VVV* group in 2 letters. (We could not, for example, use both BAAAA and BEAAA.) Despite the availability of 20 consonants, since there are only 6 vowels in the alphabet we can form only 6 *CV* groups differing each from all the other *CV* groups in 2 letters. We can, for example, combine AAA with BA, CE, DI, FO, GU, HY, but with no more *CV* groups. Having now associated 6 *CV* groups with AAA, we can associate 6 others with AAE, say BE, CI, DO, FU, GY, HA, and so on with our remaining *VVV* groups. Thus we may obtain $6\times216=1,296$ groups of the type *CVVVV*. Since our single consonant can occupy any one of 5 positions in the word, the grand total of $1C/4V$ words will be $5\times1,296=6,480$.

(3) It is not possible to increase this number. It is true that we have used only 6 different consonants in all, but the remaining 14 consonants are of no value, for if we use them to replace the single consonant already used in the words we have formed, we shall obtain new words differing in only one letter from those we already have. If, for example, we have BAAAA, CAAAE, DAAAI, FAAAO, GAAAU, HAAAY, we cannot use JAAAA, KAAAE, LAAAI, MAAAO, NAAAU or PAAAY. The case is the same no matter what position in the word the single consonant occupies. And it is manifestly impossible to add to our words by additional vowel variations as those variations have already been exhausted.

(4) It is important to note in the foregoing explanation that the number of permutations of the complete set of 20 consonants suffers a serious reduction as a result of the association of the consonants with a much more limited number of vowels. It is clear, in fact, that if there were but six consonants available instead of 20 the total number of Class II words would still remain the same, 6,480. Fourteen consonants are wholly valueless in composing these words.

(5) Perhaps a concrete example employing a miniature alphabet will be useful in demonstrating this point conclusively. For this purpose, let us take an alphabet composed of three vowels, A, E, I, and of five consonants, B, C, D, F, G, to form 4-letter words with a 2-letter difference. If we require that all four letters be vowels, we shall be able to form, with three vowels, $3\times3\times3\times1=27$ 4-letter words with a 2-letter difference. If we set up a further limitation and demand that the first three letters be vowels and the last a consonant (*VVVC*), we can produce no more words than when all four letters were vowels, viz., 27, because, despite the availability of five different consonants, we are unable to make use of them all, as will now be shown.

(6) Let us assume that our *VVVC* words are divided into two sections, *VV* and *VC*. For *VV* we have $3\times3=9$ possibilities. For *VC* we have $3\times5=15$, if we are forming only 1-letter difference words; but, as we have already seen, in order to obtain a 2-letter difference we must "sacrifice" one factor, that is, reduce it to unity. Now we cannot sacrifice the vowel of the *VC* section with its factor value of 3, and keep the consonant with its factor value of 5, because we are unable to use the full factor value of the consonant *unless the consonant has*

another letter with a factor value of 5 with which it can be permutatively associated. Since under the conditions of the problem this consonant must be associated with vowels, of which there are only three, our single consonant loses its factor value of 5 and becomes reduced to 3. It is now immaterial whether the factor that must be "sacrificed" in making the final calculation be considered as that pertaining to the vowel or to the consonant: the full number of words is $3\times3\times3\times1=27$. We may actually form a set of *VVVC* words under the conditions given:

AAAB	EAAC	IAAD	AEAC	EEAD	IEAB	AIAD	EIAB	IIAC
AAEC	EAED	IAEB	AEED	EEEB	IEEC	AIEB	EIEC	IIED
AAID	EAIB	IAIC	AEIB	EEIC	IEID	AIIC	EIID	IIIB

We have here our full quota of 27 words, and have not used the consonants F and G at all. We can use them if we will, but only to replace B, C, or D, and that will give us merely *alternative* words, not additional ones. (For example, we might have AIAG or IEAF, but each of these will show only a 1-letter difference from some words we already have.)

(7) In computing the total number of possible words with a 2-letter difference, composed of a mixture of vowels and consonants (always assuming that the consonants outnumber the vowels in the alphabet employed) if the words have one consonant each, the factor value of the set of consonants is exactly the same as that of the set of vowels with which the consonants must be associated; and then, as regards the factor which must be reduced to unity or "sacrificed," it is immaterial which is the one considered to have been sacrificed, that pertaining to the vowels or that pertaining to the consonants. If the words contain more than one consonant each, one of the factors pertaining to the consonant positions must be reduced to unity while all other factors will retain their full value.

(8) From the foregoing this general rule may be stated: In a set of unequal factors which are used as a basis for computing the total number of d-letter difference words that may be constructed, there must be at least d factors of equal and maximum value, and $(d-1)$ of them must be reduced to unity while all other factors retain their full value.

(9) In Class III, the $2C/3V$ class, we can form 10 subclasses, according to the positions occupied by the 2 consonants—*CCVVV, CVCVV, CVVCV, CVVVC, VCCVV, VCVCV, VCVVC, VVCCV, VVCVC, VVVCC.* It is obvious that each of these subclasses will yield the same number of words. Applying the general formula derived above to any of the foregoing ten subclasses, say the subclass *CCVVV*, we have $20\times1\times6\times6\times6=4320$ words for this subclass. Since there are 10 subclasses to each of which the same formula is applicable, we obtain a grand total of $10\times4{,}320=43{,}200$ words of the $2C/3V$ class.

(10) Proceeding with Class IV as with Class III, we find that there are again 10 subclasses, corresponding to the 10 possible positions of the consonants. Taking any subclass, for example, *CCCVV*, and applying the general formula

we have $20\times20\times1\times6\times6=14{,}400$ words. For 10 subclasses we have $10\times14{,}400=144{,}000$ words.

(11) Similarly Class V ($4C/1V$) will yield $20\times20\times20\times1\times6=48{,}000$ words for each position of the single vowel, or a total of $5\times48{,}000=240{,}000$ words for the entire Class.

(12) Class VI ($5C$) will give $20\times20\times20\times20\times1=160{,}000$ words.

2. We are ready now to study Table II with a view to finding an answer to the question posited: What is the theoretical maximum number of 5-letter code words that can be constructed with a minimum 2-letter difference, each code word containing a minimum of two vowels?

3. Three hypotheses may be considered with respect to how we can obtain from Table II the maximum number of code words conforming to the foregoing specifications. They are:

1st hypothesis: The maximum number may be obtained by taking *all* the words of any one class.

2nd hypothesis: The maximum number may be obtained by taking *all* the words of two or more classes without introducing any "conflicts," i.e., without violating the 2-letter differential in the case of even a single pair of words.

3rd hypothesis: The maximum number may be obtained by taking words from two or more classes, the words being selected in such a way that no conflicts will be introduced.

4. Classes V and VI can immediately be eliminated from consideration, for they do not conform to the requirement that each word contain at least two vowels. There are left for consideration, therefore, only Classes I to IV, inclusive.

5. It is obvious that if two or more of the four classes remaining for consideration can be found to conform to the requirements of the second hypothesis, we shall obtain more words than can be obtained by an adherence to the first hypothesis. Let us begin, therefore, with the class which by itself gives the greatest total number of words, viz., Class IV. Is it possible to combine Class IV words with *all* the words of any of the other three classes, without introducing any conflicts? Let us try to combine Class IV with the class showing the next greatest total number of words, viz., Class III. Let us take subclass 23 of Class IV, of the form *VCCVC* (see Table I); can we combine words of this subclass with words of subclass 9 of Class III, of the form *VCVVC*, without conflict? To be even more specific, can words of subclass 23, as exemplified in the word ABBAB be used without conflict with words of subclass 9, as exemplified in the word ABAAB? The answer must be in the negative, because as these two words now stand they show only a 1-letter difference. No matter what the specific constitution of any *VCCVC* word, it is bound to conflict with some *VCVVC* word, if *all* the words of both subclasses are taken. This is true with respect to all subclasses of Classes III and IV; hence these two classes, each taken in its entirety, cannot be used together without conflict.

6. Can words of Classes II and IV be used together without conflict? Fol-

lowing the same reasoning as in Paragraph 5, it will become apparent that they can. For, let us take any subclass of Class IV, subclass 23 for example, *VCCVC*, and combine it with any subclass of Class II, subclass 5 for example, *VCVVV*; no conflicts can arise because the third and the fifth positions in these subclasses will always be occupied in the one case by two consonants, in the other case, by two vowels. Take another example: combine subclass 3, *VVVCV*, with subclass 25, *VCVCC*; again no conflicts can arise because the second and fifth positions in these two subclasses will always be occupied in the one case by two vowels, in the other, by two consonants. This holds for all the subclasses of Classes II and IV when combined. Hence, the two classes can be used together without conflict, yielding a maximum total of 6,480+144,000=150,480 words.[1]

7. It is obvious that words of Class I cannot be included with words of Classes IV and II without producing conflicts, since words of Classes I and II will show conflicts between themselves.

8. The only other classes that might be employed together in their entireties are Classes I and III, but they will yield a total of only 44,496 words. If, then, the maximum number of words is to be obtained by the association of complete classes, it will be by employing Classes II and IV in complete series, yielding a total of 150,480 words.

9. We are, however, not justified in calling this number the *maximum* number possible unless and until we can dispose of the third hypothesis set forth in paragraph 3: can the maximum number be obtained by taking *some* words from one class and adding *some* words from the other classes, the words being selected in stch a way that no words will conflict with one another? Let us see.

10. We have already associated Classes II and IV. We cannot add to the total number of words by taking any words from Class I. All that any Class I words can do is to *replace* words of Class II, and that will not add to the total. As a matter of fact, it will decrease it. With Class III, however, the case is different. In forming words of Class II we use, or need use, only six different consonants. Just as the words of Class I are formed by associations of one or another of six different vowels in each of the five positions, so the words of Class II use the same six vowels in four of the five positions, and six different consonants in the remaining position. Let us assume that the six different consonants used are BCDFGH.

11. In the two positions in the words of Class III where consonants are used, *all* the consonants are employed. As a consequence we can add a certain number of words from Class III without conflicting with any of the words of Class II. Thus, we shall have in Class II, say, the word AEIOB. We can now take AEIJK, AEIKL, AEILM, AEIMN, etc., from Class III, and still preserve the 2-letter differential between the words of Class II and those of Class III.

12. There are fourteen consonants not employed in words of Class II. These

[1] Credit for the association of Class IV with Class II to produce the number 150,480 is due to Dr. L. H. Canfield, of the College of the City of New York.

14 consonants will form $6\times6\times6\times14=3{,}024$ combinations, which when multiplied by the ten possible positions of the two consonants in each word will yield 30,240 Class III words. All these words can be used with the complete series of words of Class II without interfering with the 2-letter differential.

13. We have, however, in addition to the complete set of words of Class II, used, in obtaining our 150,480 words, a complete set of words of Class IV. Before concluding that the 30,240 words just obtained can be added to our total, we must see whether they or any of them conflict with the Class IV words.

14. Both Class III and Class IV, at those places in their words where they employ consonants, use, for the complete set of words, the full number of twenty consonants. The only difference between a word of Class III and one of Class IV is that at one point where a word of Class III has a vowel a word of Class IV will have a consonant. It is therefore impossible to add words of Class III to a complete set of words of Class IV. We can *substitute* a word of Class III for one of Class IV, but this is an even exchange and will not increase our word total. If, then, we increase our word total by adding, to the words of Class II, non-conflicting words from Class III, we lose one word from Class IV for each word so added, and at the end we are exactly where we were before.

15. One possibility remains: it might be possible to omit *some* of the words of Class II or Class IV and substitute for each word so omitted more than one word from Class I or Class III. But if we replace any word of Class II by a word of Class I, we lose five words in Class II for each one that is taken for such replacement in Class I. And if we replace any word of Class II by a word of Class III, we shall gain ten words in Class III for every five so replaced in Class II—but we shall lose ten words of Class IV for every ten words taken from Class III. That is, for every ten words gained, 15 are lost as a result of the substitution.

16. It has been shown that we can obtain 150,480 words by using all the words of Class II and all those of Class IV. This total cannot be increased by adding words of Class I or Class III. Neither can it be increased by dropping some of the words of Class II or Class IV and adding a greater number of Class I or Class III. That is, it cannot be increased at all, and is the maximum number obtainable.

Query II

The 2-letter difference code words of modern 5-letter codes are usually compiled by reference to tables known under various designations, such as "Permutation Table," "Mutilation Chart," "Error Detector Chart," etc. An example of a typical table is shown in Table III below. We shall refer to such a table as a code-word construction table.

The code words afforded by the foregoing table are formed from the table by combining three elements: (1) a pair of letters from section 1, (2) a single letter from section 2, and (3) a pair of letters from section 3; and the combination must be made according to the rule that the initial pair and the middle letter must lie in the same vertical line (extended), the middle letter and the final pair must lie in the same horizontal line (extended).

Section 1

aa	ab	ac	ad	ae	af	ag	ah	ai	aj	ak	al	am	an	ao	ap	aq	ar	as	at	au	av	aw	ax	ay	az
bb	bc	bd	be	bf	bg	bh	bi	bj	bk	bl	bm	ba	bo	bp	bq	br	bs	bt	bu	bv	bw	bx	by	bz	bn
cc	cd	ce	cf	cg	ch	ci	cj	ck	cl	cm	ca	cb	cp	cq	cr	cs	ct	cu	cv	cw	cx	cy	cz	cn	co
dd	de	df	dg	dh	di	dj	dk	dl	dm	da	db	dc	dq	dr	ds	dt	du	dv	dw	dx	dy	dz	dn	do	dp
ee	ef	eg	eh	ei	ej	ek	el	em	ea	eb	ec	ed	er	es	et	eu	ev	ew	ex	ey	ez	en	eo	ep	eq
ff	fg	fh	fi	fj	fk	fl	fm	fa	fb	fc	fd	fe	fs	ft	fu	fv	fw	fx	fy	fz	fn	fo	fp	fq	fr
gg	gh	gi	gj	gk	gl	gm	ga	gb	gc	gd	ge	gf	gt	gu	gv	gw	gx	gy	gz	gn	go	gp	gq	gr	gs
hh	hi	hj	hk	hl	hm	ha	hb	hc	hd	he	hf	hg	hu	hv	hw	hx	hy	hz	hn	ho	hp	hq	hr	hs	ht
ii	ij	ik	il	im	ia	ib	ic	id	ie	if	ig	ih	iv	iw	ix	iy	iz	in	io	ip	iq	ir	is	it	iu
jj	jk	jl	jm	ja	jb	jc	jd	je	jf	jg	jh	ji	jw	jx	jy	jz	jn	jo	jp	jq	jr	js	jt	ju	jv
kk	kl	km	ka	kb	kc	kd	ke	kf	kg	kh	ki	kj	kx	ky	kz	kn	ko	kp	kq	kr	ks	kt	ku	kv	kw
ll	lm	la	lb	lc	ld	le	lf	lg	lh	li	lj	lk	ly	lz	ln	lo	lp	lq	lr	ls	lt	lu	lv	lw	lx
mm	ma	mb	mc	md	me	mf	mg	mh	mi	mj	mk	ml	mz	mn	mo	mp	mq	mr	ms	mt	mu	mv	mw	mx	my
no	np	nq	nr	ns	nt	nu	nv	nw	nx	ny	nz	nn	nb	nc	nd	ne	nf	ng	ni	nh	nj	nk	nl	nm	na
op	oq	or	os	ot	ou	ov	ow	ox	oy	oz	on	oo	oc	od	oe	of	og	oh	oj	oi	ok	ol	om	oa	ob
pq	pr	ps	pt	pu	pv	pw	px	py	pz	pn	po	pp	pd	pe	pf	pg	ph	pi	pk	pj	pl	pm	pa	pb	pc
qr	qs	qt	qu	qv	qw	qx	qy	qz	qn	qo	qp	qq	qe	qf	qg	qh	qi	qj	ql	qk	qm	qa	qb	qc	qd
rs	rt	ru	rv	rw	rx	ry	rz	rn	ro	rp	rq	rr	rf	rg	rh	ri	rj	rk	rm	rl	ra	rb	rc	rd	re
st	su	sv	sw	sx	sy	sz	sn	so	sp	sq	sr	ss	sg	sh	si	sj	sk	sl	sa	sm	sb	sc	sd	se	sf
tu	tv	tw	tx	ty	tz	tn	to	tp	tq	tr	ts	tt	th	ti	tj	tk	tl	tm	tb	ta	tc	td	te	tf	tg
uv	uw	ux	uy	uz	un	uo	up	uq	ur	us	ut	uu	ui	uj	uk	ul	um	ua	uc	ub	ud	ue	uf	ug	uh
vw	vx	vy	vz	vn	vo	vp	vq	vr	vs	vt	vu	vv	vj	vk	vl	vm	va	vb	vd	vc	ve	vf	vg	vh	vi
wx	wy	wz	wn	wo	wp	wq	wr	ws	wt	wu	wv	ww	wk	wl	wm	wa	wb	wc	we	wd	wf	wg	wh	wi	wj
xy	xz	xn	xo	xp	xq	xr	xs	xt	xu	xv	xw	xx	xl	xm	xa	xb	xc	xd	xf	xe	xg	xh	xi	xj	xk
yz	yn	yo	yp	yq	yr	ys	yt	yu	yv	yw	yx	yy	ym	ya	yb	yc	yd	ye	yg	yf	yh	yi	yj	yk	yl
zn	zo	zp	zq	zr	zs	zt	zu	zv	zw	zx	zy	zz	za	zb	zc	zd	ze	zf	zh	zg	zi	zj	zk	zl	zm

b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	a
c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	a	b
d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	a	b	c
e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	a	b	c	d
f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	a	b	c	d	e
g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	a	b	c	d	e	f
h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	a	b	c	d	e	f	g
i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h
j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i
k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	j
l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	j	k
m	n	o	p	q	r	s	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	j	k	l
n	o	p	q	r	s	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	j	k	l	m
o	p	q	r	s	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	j	k	l	m	n
p	q	r	s	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o
q	r	s	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p
r	s	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q
s	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r
t	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s
u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t
v	w	x	y	z	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u
w	x	y	z	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v
x	y	z	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w
y	z	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x
z	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y
a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z

Section 2

az
bn
co
dp
eq
fr
gs
ht
iu
jv
kw
lx
my
na
ob
pc
qd
re
sf
tg
uh
vi
wj
xk
yl
zm

TABLE III

An Example of a Code Word Construction Table.

Reproduced by permission of the Code Compiling Co., Inc., New York, publishers of the Universal Trade Code, 1921. All the cells in Section 1 and Section 3, many of which are vacant in the original table, have been filled in with the proper pairs of letters by the present authors, to meet the special requirements of this topic.

aa	bb	cc	dd	ee	ff	gg	hh	ii	jj	kk	ll	mm	nz	on	po	qp	rq	sr	ts	ut	vu	wv	xw	yx	zy
am	ba	cb	dc	ed	fe	gf	hg	ih	ji	kj	lk	ml	ny	oz	pn	qo	rp	sq	tr	us	vt	wu	xv	yw	zx
al	bm	ca	db	ec	fd	ge	hf	ig	jh	ki	lj	mk	nx	oy	pz	qn	ro	sp	tq	ur	vs	wt	xu	yv	zw
ak	bl	cm	da	eb	fc	gd	he	if	jg	kh	li	mj	nw	ox	py	qz	rn	so	tp	uq	vr	ws	xt	yu	zv
aj	bk	cl	dm	ea	fb	gc	hd	ie	jf	kg	lh	mi	nv	ow	px	qy	rz	sn	to	up	vq	wr	xs	yt	zu
ai	bj	ck	dl	em	fa	gb	hc	id	je	kf	lg	mh	nu	ov	pw	qx	ry	sz	tn	uo	vp	wq	xr	ys	zt
ah	bi	cj	dk	el	fm	ga	hb	ic	jd	ke	lf	mg	nt	ou	pv	qw	rx	sy	tz	un	vo	wp	xq	yr	zs
ag	bh	ci	dj	ek	fl	gm	ha	ib	jc	kd	le	mf	ns	ot	pu	qv	rw	sx	ty	uz	vn	wo	xp	yq	zr
af	bg	ch	di	ej	fk	gl	hm	ia	jb	kc	ld	me	nr	os	pt	qu	rv	sw	tx	uy	vz	wn	xo	yp	zq
ae	bf	cg	dh	ei	fj	gk	hl	im	ja	kb	lc	md	nq	or	ps	qt	ru	sv	tw	ux	vy	wz	xn	yo	zp
ad	be	cf	dg	eh	fi	gj	hk	il	jm	ka	lb	mc	np	oq	pr	qs	rt	su	tv	uw	vx	wy	xz	yn	zo
ac	bd	ce	df	eg	fh	gi	hj	ik	jl	km	la	mb	no	op	pq	qr	rs	st	tu	uv	vw	wx	xy	yz	zn
ab	bc	cd	de	ef	fg	gh	hi	ij	jk	kl	lm	ma	nn	oo	pp	qq	rr	ss	tt	uu	vv	ww	xx	yy	zz
az	bn	co	dp	eq	fr	gs	ht	iu	jv	kw	lx	my	na	ob	pc	qd	re	sf	tg	uh	vi	wj	xk	yl	zm
ay	bz	cn	do	ep	fq	gr	hs	it	ju	kv	lw	mx	nm	oa	pb	qc	rd	se	tf	ug	vh	wi	xj	yk	zl
ax	by	cz	dn	eo	fp	gq	hr	is	jt	ku	lv	mw	nl	om	pa	qb	rc	sd	te	uf	vg	wh	xi	yj	zk
aw	bx	cy	dz	en	fo	gp	hq	ir	js	kt	lu	mv	nk	ol	pm	qa	rb	sc	td	ue	vf	wg	xh	yi	zj
av	bw	cx	dy	ez	fn	go	hp	iq	jr	ks	lt	mu	nj	ok	pl	qm	ra	sb	tc	ud	ve	wf	xg	yh	zi
au	bv	cw	dx	ey	fz	gn	ho	ip	jq	kr	ls	mt	nh	oi	pj	qk	rl	sm	ta	ub	vc	wd	xe	yf	zg
at	bu	cv	dw	ex	fy	gz	hn	io	jp	kq	lr	ms	ni	oj	pk	ql	rm	sa	tb	uc	vd	we	xf	yg	zh
as	bt	cu	dv	ew	fx	gy	hz	in	jo	kp	lq	mr	ng	oh	pi	qj	rk	sl	tm	ua	vb	wc	xd	ye	zf
ar	bs	ct	du	ev	fw	gx	hy	iz	jn	ko	lp	mq	nf	og	ph	qi	rj	sk	tl	um	va	wb	xc	yd	ze
aq	br	cs	dt	eu	fv	gw	hx	iy	jz	kn	lo	mp	ne	of	pg	qh	ri	sj	tk	ul	vm	wa	xb	yc	zd
ap	bq	cr	ds	et	fu	gv	hw	ix	jy	kz	ln	mo	nd	oe	pf	qg	rh	si	tj	uk	vl	wm	xa	yb	zc
ao	bp	cq	dr	es	ft	gu	hv	iw	jx	ky	lz	mn	nc	od	pe	qf	rg	sh	ti	uj	vk	wl	xm	ya	zb
an	bo	cp	dq	er	fs	gt	hu	iv	jw	kx	ly	mz	nb	oc	pd	qe	rf	sg	th	ui	vj	wk	xl	ym	za

Section 3

Given a specific, completely-filled code-word construction table for constructing 5-letter words with a 2-letter difference, this question now arises: How many words are there of each of the six classes discussed under Query I?

Solution

1. The initial step in the solution to this query consists in determining by actual count the number of words of the $5V$ class in the table under investigation. Let us assume that in a certain table this was found to be 236 words.

2. Words of the class $4V/1C$ comprise five subclasses, according to the position of the consonant, (see Table I). Each of these subclasses will yield 1,296 2-letter difference words. Any word of the class $VVVVV$ will eliminate one word from each of the foregoing subclasses of $4V/1C$ words. For example, AEIOU would eliminate AEIOC, AEICU, AECOU, ACIOU, and CEIOU. If then, we use 236 five-vowel words, we shall have remaining $1,296-236=1,060$ words in each of the five subclasses comprising the class $4V/1C$, making a total of $5\times1,060=5,300$ words.

3. Words of the class $3V/2C$ comprise ten subclasses, according to the positions of the two consonants (see Table I). Each of these subclasses will yield 4,320 2-letter difference words. Certain subclasses of class $4V/1C$ will conflict with certain subclasses of class $3V/2C$. For example, words of subclass 2, such as AEIOR, will conflict with words of subclass 7, such as AEIQR, as well as with words of subclass 8, such as AEQOR, of subclass 9, such as AQIOR, and of subclass 10, such as QEIOR. Thus, each word of subclass 2 will eliminate *four* words from class $3V/2C$ words. Since there are 1,060 words of subclass 2 ($VVVVC$) and 4,320 words in each of the subclasses 7, 8, 9 and 10, (see Table I) we shall have remaining in each of these four last-named subclasses the difference between 4,320 and 1,060, which is 3,260 words. Applying the same reasoning to all the subclasses of class II, we draw up the following table:

TABLE IV

Subclass	2—*VVVVC* conflicts with	Subclass	3—*VVVCV* conflicts with
"	7—*VVVCC*, with	"	7—*VVVCC*, with
"	8—*VVCVC*, with	"	11—*VVCCV*, with
"	9—*VCVVC*, and with	"	12—*VCVCV*, and with
"	10—*CVVVC*	"	13—*CVVCV*
Sublcass	4—*VVCVV* conflicts with	Subclass	5—*VCVVV* conflicts with
"	8—*VVCVC*, with	"	9—*VCVVC*, with
"	11—*VVCCV*, with	"	12—*VCVCV*, with
"	14—*VCCVV*, and with	"	14—*VCCVV*, and with
"	15—*CVCVV*	"	16—*CCVVV*

Subclass	6—*CVVVV* conflicts with
"	10—*CVVVC*, with
"	13—*CVVCV*, with
"	15—*CVCVV*, and with
"	16—*CCVVV*

4. Each of the ten subclasses 7 to 16, inclusive, is here duplicated in conflicts; but we need, of course, to eliminate words only once, in order to dispose of the conflicts with retained words of subclasses 2 to 6, inclusive. Therefore, we shall have remaining in each of subclasses 7 to 16, inclusive, 4,320−1,060 or 3,260 words, and hence the total number remaining, after deduction for conflicts, for the whole of class III is 3,260×10 or 32,600 words.

5. If we continue this process with the remaining classes of words, the following final results are obtained, shown in condensed mathematical form:

TABLE V

Class	*Description*	*Calculation*	*No. of Words*
I	$5V$	(as found by actual count in a specific table) =	236
II	$4V/1C$	= 1,296− 236= 1,060; 1,060× 5 =	5,300
III	$3V/2C$	= 4,320− 1,060= 3,260; 3,260×10 =	32,600
IV	$2V/3C$	= 14,400− 3,260= 11,140; 11,140×10 =	111,400
V	$1V/4C$	= 48,000−11,140= 36,860; 36,860× 5 =	184,300
VI	$5C$	=160,000−36,860=123,140	123,140
		Grand total	456,976

6. It will be noted that the grand total in the foregoing calculation checks, since $26^4=456{,}976$. In similar calculations applicable to any other code-word construction table, while the individual totals may vary (according to the number of $5V$ words formed by the table), the grand total for a completely filled construction table must obviously be the same as that obtained above, viz., 456,976 words.

Query III

Experience has shown that if a code contain two such code words as, for example, ABCDE and BACDE, confusion may arise from the accidental transposition (in writing or telegraphing) of the letters A and B. It has accordingly been found advisable to construct codes so that no two code words differing from each other merely in the transposition of two adjacent letters will be included in the same code. A code-word construction table affording code words which will show no transpositions of adjacent letters can, however, be made when the number of different letters, λ, used in its construction is odd. The English alphabet contains 26 letters. To drop one letter in order to make λ odd would reduce the total number of words available. We may, however, add an extra character to the alphabet, giving $(\lambda+1)$ characters, construct a table without transpositions, and then eliminate all words containing the extra character. This will leave only words containing the λ letters, and these will contain no transpositions. We shall, however, lose a certain number of words that can be made from λ letters. How many shall we lose?

Solution

1. We may experiment with the miniature $(\lambda+1)$ table below, where the extra character, added to the alphabet to make the λ letter alphabet a $(\lambda+1)$ letter alphabet, is represented by the asterisk:

TABLE VI

Section I

1st and 2nd letters							
AA	AB	AC	AD	AE	AF	A*
BB	BC	BD	BE	BF	B*	BA
CC	CD	CE	CF	C*	CA	CB
DD	DE	DF	D*	DA	DB	DC
EE	EF	E*	EA	EB	EC	ED
FF	F*	FA	FB	FC	FD	FE
**	*A	*B	*C	*D	*E	*F

($\lambda = 6$)

3rd letter							
A	B	C	D	E	F	*
B	C	D	E	F	*	A
C	D	E	F	*	A	B
D	E	F	*	A	B	C
E	F	*	A	B	C	D
F	*	A	B	C	D	E
*	A	B	C	D	E	F

Section II

Section III	4th and 5th letters						
	AA	BB	CC	DD	EE	FF	**
	BA	CB	DC	ED	FE	*F	A*
	CA	DB	EC	FD	*E	AF	B*
	DA	EB	FC	*D	AE	BF	C*
	EA	FB	*C	AD	BE	CF	D*
	FA	*B	AC	BD	CE	DF	E*
	*A	AB	BC	CD	DE	EF	F*

2. Examination of this miniature table of $(\lambda+1)$ characters, where λ is 6, will show that it can yield words containing only the original λ letters[1] as follows:

(1) One set of words having $\lambda^2+\lambda(\lambda-1)(\lambda-1)$ words
(2) $(\lambda-1)$ sets of words each having $\lambda(\lambda-1)+(\lambda-1)(\lambda-1)(\lambda-1)$ words
(3) One set of words having $\lambda(\lambda-1)(\lambda-1)$ words

Adding all words we have:

$$\lambda^4 - \lambda^3 + \lambda^2 - \lambda + 1 \text{ or } \lambda^4 - (\lambda^2 + 1)(\lambda - 1)$$

3. Now λ letters arranged in a construction table will give λ^4 2-letter difference words. Hence, by adding the extra character to the table and eliminating words in which the extra character appears, we shall lose $(\lambda^2+1)(\lambda-1)$ words.

4. Accordingly, if $\lambda = 26$, we shall lose from the complete table for 26 letters $(26^2+1)25 = 677 \times 25 = 16{,}925$ words. This leaves $456{,}976 - 16{,}925 = 440{,}051$ words.

5. It may be interesting to know how many words will be eliminated by the process described, from the complete table based on $(\lambda+1)$ letters. This table will yield $(\lambda+1)^4$ words, which equals $\lambda^4+4\lambda^3+6\lambda^2+4\lambda+1$ words. As already shown, if we omit words containing the extra character, we shall have $\lambda^4-\lambda^3+\lambda^2-\lambda+1$ words. Hence

[1] One special case must be considered. The added letter will appear twice in each column of Section I and twice in each row of Section III, except that there will be one column and one row respectively where it will appear only once. Should the arrangement of the code-word construction table as a whole be such that this one column and one row are not associated in forming words, one additional word will be lost. The deduction of the words from the total for λ letters will then be $(\lambda^2+1)(\lambda-1)+1$ and the deduction from the total for $\lambda+1$ letters will be $5\lambda^3+5\lambda^2+5\lambda+1$.

$$(\lambda + 1)^4 - (\lambda^4 - \lambda^3 + \lambda^2 - \lambda + 1) = 5\lambda^3 + 5\lambda^2 + 5\lambda$$

words will be eliminated.
For 27 characters the net total would therefore be:

$$27^4 - [5(27^3) + 5(27^2) + 5(27)] = 531,441 - (87,880 + 3380 + 130)$$
$$= 531,441 - 91,390 = 440,051.$$

6. It may further be interesting to see what is gained by this process over the simpler method of constructing a table with $(\lambda-1)$ letters. This table would give $(\lambda-1)^4=\lambda^4-4\lambda^3+6\lambda^2-4\lambda+1$ words. The table containing $\lambda+1$ letters gives, as we have seen, when words containing the extra letters are rejected, $\lambda^4-(\lambda^2+1)(\lambda-1)$ words. Subtracting $\lambda^4-4\lambda^3+6\lambda^2-4\lambda+1$ from the latter quantity, we have $3\lambda^3-5\lambda^2+3\lambda$ as the difference between the respective numbers of words yielded by the two tables. If $\lambda=26$, a table of $(\lambda-1)$ or 25 letters will give 390,625 words, while a table of $(\lambda+1)$ or 27 letters, omitting words containing the extra letter, will give 440,051, as has been shown.

Note

So far as words of Category B are concerned, since no limitations are placed upon their composition by the present regulations, the total number of code words with a 2-letter difference available for code compilers and code users is 26^4 or 456,976 words. If nontransposability of adjacent letters referred to in the preceding section is taken into consideration in the elaboration of the construction table, this total becomes reduced to either 440,051 words or 390,625 words, depending upon the method selected.

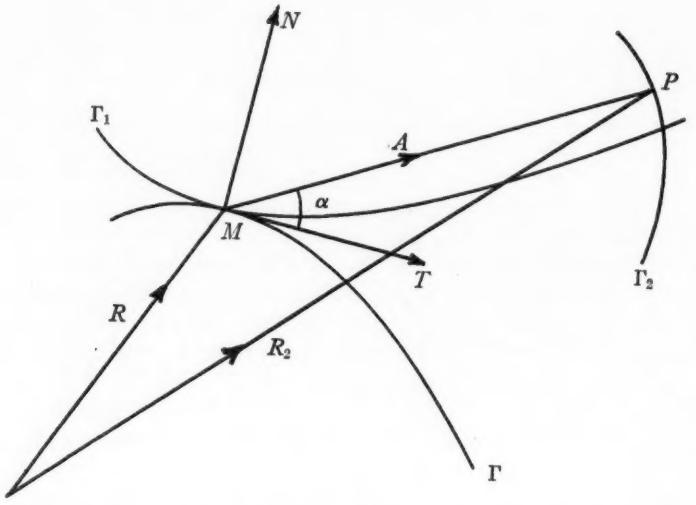
CYCLOIDAL CURVES

By SOLOMON BILINSKY, Washington University

When one plane curve rolls upon another, every point fixed relative to the rolling curve and in its plane describes a new curve. In the particular instance where the fixed curve is a straight line and the rolling curve a circle every point on the circle describes an ordinary cycloid. Other particular instances are quite as well known. It is the purpose of this discussion, however, to treat the general case, and to demonstrate the simplicity and straightforwardness with which vector analysis provides a means for the study of such curves, which will in the general case be designated as cycloidal curves.[1]

We suppose that the curve Γ_1 rolls upon the fixed curve Γ and that the point P, fixed relative to Γ_1, describes a curve Γ_2. As Γ_1 rolls along Γ the tangents and

[1] A different treatment of this subject may be found in A. Mannheim, *Géométrie Descriptive*, 2nd edition (1886), pp. 175–180.



normals to each curve are coincident and equal lengths of arc are measured off on each. We will agree that the unit normal N is to be drawn from the point of contact M so that κ_1, the curvature of Γ_1 is always positive, in which case the sign of κ, the curvature of Γ, will be positive or negative according as the concavity of Γ is in the same or opposite direction to that of Γ_1. The unit tangent T is to be drawn in the direction of increasing arc-length. Let A be a unit vector making an angle α with T and drawn so that aA is the vector MP.

It will first be necessary to find explicit expression[1] for a and the first derivatives of both a and A. Since this does not entirely involve the problem of rolling curves we will assume for the moment that Γ_1 is fixed and that the extremity of the vector R, drawn from a chosen origin, runs along Γ_1. Let the point P be fixed by the vector R_P, drawn from the same origin as R. We have

$$R_P = R + aA, \tag{1}$$

$$A = T\cos\alpha + N\sin\alpha, \tag{2}$$

and, in addition, a unit vector B, perpendicular to A:

$$B = N\cos\alpha - T\sin\alpha. \tag{3}$$

By scalar multiplication with T, we have from (2) and (3)

$$T\cdot B = -\sin\alpha \quad T\cdot A = \cos\alpha. \tag{3.1}$$

[1] The values of a and its derivatives obviously depend only upon the form of the curve Γ_1. A is determined by its orientation with respect to T and N which are coincident at M. The derivatives of A, however, depend upon changes in T and N, and these latter depend only upon the curvature of Γ.

Denoting differentiation with respect to s_1, the arc length of Γ_1, by an accent and subscript 1, we have from (2) and (3)

$$A_1' = \nu_1 B, \quad \nu_1 = \alpha_1' + \kappa_1;$$

from (1)

$$0 = T + a_1' A + a(\nu_1 B).$$

By scalar multiplication of the last equation with A and B in turn, we have

$$\sin\alpha = a\nu_1 \quad \cos\alpha = -a_1', \tag{3.2}$$

in view of relations (3.1).

Curve Γ_1 is now allowed to roll along Γ and the terminus of R kept on the point of contact. Denoting differentiation with respect to s, the arc length of Γ, which is the same as s_1, by an accent, we have from (2) and (3), as before

$$A' = \nu B, \quad \nu = \alpha' + \kappa. \tag{3.3}$$

Let the locus of P be determined by the vector R_2, drawn from the origin of R. Then

$$R_2 = R + aA.$$

Differentiating with respect to s:

$$T_2 s_2' = T + a'A + a\nu B. \tag{4}$$

The scalar product of the last equation with A yields

$$T_2 \cdot A s_2' = \cos\alpha - \cos\alpha = 0,$$

by (3.1) and (3.2), with the observation that $a_1' = a'$.

It follows that T_2 is perpendicular to A; a fact that is otherwise evident from the consideration that M is the instantaneous center of rotation. We shall write $T_2 \equiv -B$, and since N_2 is perpendicular to T_2, $N_2 \equiv -A$. The choice of sign is arbitrary and only serves to fix the directions along which s_2 and κ_2 are measured.

We have now

$$T_2' = -B',$$

or

$$\kappa_2 s_2' N_2 = \nu A,$$

the last member being obtained from (3). Since $N_2 \equiv -A$,

$$\kappa_2 = -\frac{\nu}{s_2'} = \frac{\nu}{a\nu - \sin\alpha},$$

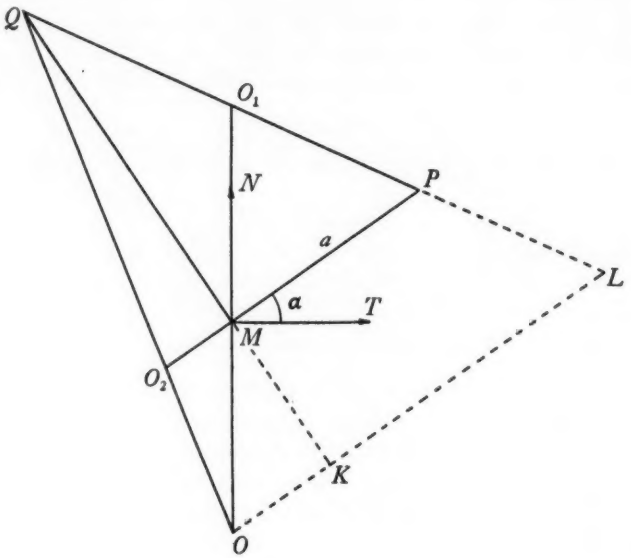
for by scalar multiplication of (4) with B, it is found that $s_2' = \sin\alpha - a\nu$. Now replacing ν and $\sin\alpha$ from the expression in (3.3) and (3.2),

$$\kappa_2 = \frac{a(\kappa - \kappa_1) + \sin\alpha}{a^2(\kappa - \kappa_1)}. \tag{5}$$

In terms of the radius of curvature ρ, (5) becomes

$$\rho_2 = \frac{a^2(\rho_1 - \rho)}{a(\rho_1 - \rho) + \rho\rho_1 \sin\alpha}. \tag{6}$$

The center of curvature of Γ_2 at P can easily be found by geometrical construction.[1] Though the method is general we shall consider for simplicity the case in which the center of curvature O_1 of Γ_1 and O of Γ are on opposite sides of the point of contact M.



Draw the lines PO_1, and MQ perpendicular to MP at M. Draw the straight line joining the point of intersection Q of these two lines with O. Then QO and PM intersect in O_2, the desired center of curvature of Γ_2 at P.

PROOF: Draw OL parallel to MP, cutting O_1P in L and QM in K. Then, by inspection of the figure, remembering that we have chosen the case for which ρ is negative, we have

$$OK = -\rho\sin\alpha, \quad \frac{OL}{a} = \frac{\rho_1 - \rho}{\rho_1}, \quad \frac{\rho_2}{a} = \frac{OL}{OL - OK};$$

and we are easily led to equation (6).

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[1] Mannheim, p. 175. The construction is here attributed to Euler and Savary.

QUESTIONS AND DISCUSSIONS

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems which are reserved for the department of Problems and Solutions.

A REMARK ON DEDEKIND CUTS

By THEODORE WEAVER and THEODORE SUCKAU, Ohio State University

There are several ways of defining real numbers. One of the most widely used is that due to Dedekind, namely the Dedekind cut in the domain of rational numbers. The question arises as to whether or not there are cuts which do not define a rational number. The commonest example of such a cut is the following:

r is in the class L if $r > 0$ and $r^2 < 2$, or if $r \leqq 0$;

r is in the class U if $r > 0$ and $r^2 > 2$.

To show that this cut does not define a rational number we must demonstrate that the class L has no largest element (number) and that the class U has no smallest element (number). The usual way to show this is to prove that given $r>0$ in L we can find h such that $0<h$ and such that $(r+h)^2<2$; also for $R>0$ in U we can find h such that $(R-h)^2>2$, and $R-h>0$.

While working on this problem in an introductory course in analysis conducted by Professor Tibor Rado at the Ohio State University we found a way of proving this which was both interesting to the class and to the instructor. We thought that it might be interesting to others because of its connection with more advanced mathematics.

We found that by properly choosing h we can obtain an expression which could be used to show not only that there is no smallest element in the upper class U, but also that there is no largest element in the lower class L. This expression is

$$\bar{r} = \frac{4 + 3r}{3 + 2r},$$

where r is any positive rational number. We find

$$\bar{r}^2 = 2 + \frac{r^2 - 2}{(3 + 2r)^2},$$

$$\bar{r}^2 - r^2 = \frac{4(2 + 3r + r^2)(2 - r^2)}{(3 + 2r)^2};$$

and hence, if r is in U, $\bar{r}$ is also in U and is $<r$; and if $r>0$ is in L, then $\bar{r}$ is also in L and is $>r$.

The reader who is familiar with the theory of linear transformations will recognise the above expression for $\bar{r}$ as a linear hyperbolic transformation with an attractive fixed point at $+\sqrt{2}$.

It is easily found that the most general such transformation with rational coefficients may be written in the form

$$\bar{r} = \frac{br + 2a}{ar + b},$$

where a and b are positive integers such that

$$\begin{vmatrix} b & 2a \\ a & b \end{vmatrix} > 0.$$

NOTES ON A DIOPHANTINE EQUATION

By H. GROSSMAN, De Witt Clinton High School, New York, N. Y.

Given the equation $ax^3+bx^2+cx+d=y^2$, where a, b, c, d are rational, $a\neq 0$, $d=\square\neq 0$, the following is an interesting device for constructing a chain of rational solutions.

If (x_i, y_i) is any solution of the original equation other than the trivial ones $(0, \pm\sqrt{d})$, it may be shown that

$$x_{i+1} = (cx_i + 2d \pm 2\sqrt{d}\, y_i)/(ax_i^2) \tag{1}$$

gives a solution (x_{i+1}, y_{i+1}).

From (1) it follows that $\sqrt{d}+cx_i/(2\sqrt{d})-ax_{i+1}x_i^2/(2\sqrt{d}) = \mp y_i$. If we square both sides, subtract from the original equation (with $x=x_i$, $y=y_i$), and divide by $x_i^2\neq 0$, we obtain

$$\left(\frac{a^2x_{i+1}^2}{4d}\right)x_i^2 - \left(\frac{acx_{i+1}}{2d} + a\right)x_i + \left(\frac{c^2}{4d} - ax_{i+1} - b\right) = 0. \tag{2}$$

The x_i-discriminant of this equation,

$$\frac{a^2c^2x_{i+1}^2}{4d^2} + \frac{a^2cx_{i+1}}{d} + a^2 - \frac{a^2x_{i+1}^2}{d}\left(\frac{c^2}{4d} - ax_{i+1} - b\right),$$

must be a rational square which we may call $a^2y_{i+1}^2/d$. Therefore

$$a^3x_{i+1}^3/d + a^2bx_{i+1}^2/d + a^2cx_{i+1}/d + a^2 = a^2y_{i+1}^2/d,$$

or multiplying by d/a^2, $ax_{i+1}^3+bx_{i+1}^2+cx_{i+1}+d=y_{i+1}^2$; i.e., x_{i+1} is also a solution of the original equation. Since x_i and x_{i+1} enter symmetrically into equation (2), it follows that if x_{i+1}, like x_i, $\neq 0$, the relation between them is reciprocal and

$$x_i = (cx_{i+1} + 2d \pm 2\sqrt{d}\, y_{i+1})/(ax_{i+1}^2).$$

Let us extend the definition (1) of x_{i+1} to the case $x_i=0$. We must take the negative determination of the double sign (if we take y_i positive) so that the numerator remains zero and the fraction does not become infinite. By double differentiation of both numerator and denominator and substitution of the limiting values of x and y, we find the limit of the fraction as x approaches 0 to be $(c^2-4bd)/(4ad)$. Since 0 is a solution of the original equation, this by continuity must also be a solution. The corresponding y is $(c^3-4bcd+8ad^2)/(8ad^{3/2})$. From this solution, by (1), we may derive the solution whose x is $8d(c^3-4bcd+8ad^2)/(c^2-4bd)^2$, etc. In general, we may construct a chain of solutions by successive applications of (1), making at each step that choice of the double sign which gives an x_{i+2} distinct from x_i (since for a definite x_i and x_{i+1} only one choice of the double sign in $x_i=(cx_{i+1}+2d\pm 2\sqrt{d}\,y_{i+1})/(ax_{i+1}^2)$ may be made), so that $x_{i+2}=x_i\pm(4\sqrt{d}\,y_{i+1})/(ax_{i+1}^2)$ (where again only one choice of the double sign may be used). The ambiguous sign makes it desirable to take only positive values of y.

The case $d\neq\square$, if we know one of its solutions (h, k) (and no trivial solution is available), may be reduced, as is well known, to the previous case by the transformation $x=X+h$. To the X of each solution of the new equation we may add h in order to obtain the x of the corresponding solution of the original equation. We may treat similarly the case $d=0$ (and $d=\square$). We can not however take $h=0$. If x_1 is a solution of $ax^3+bx^2+cx=y^2$, c/ax_1 (suggested by (1)) may be shown to be a solution also.

The equation $ax^3+bx^2y+cxy^2+dy^3=z^2$ may be solved by setting $y=d$ or any other positive or negative odd power of d (or similarly setting $x=a$ or any other positive or negative odd power of a), thus reducing to the first case.

For the equation $x^3+x^2+x+1=y^2$ (Dickson's *History of the Theory of Numbers*, Vol. I, p. 56), the method of this paper gives the following two chains of solutions:

$(0, 1), (-3/4, 5/8), (40/9, 287/27), (561/400, 21359/8000), \cdots$

$(-1, 0), (1, 2), (7, 20), (-31/49, 246/343), (-161/961, 27560/29791), \cdots$

If $f(x)=x^3+x^2+x+1=(1+x)\ (1+x^2)=\square$, then $f\{(1-x)/(1+x)\}=4(1+x^2)/(1+x)^3=\square$, so that if x is a solution, $(1-x)/(1+x)$ is also a solution, the two being in reciprocal relation. This relation exists between each solution in the first chain and a corresponding solution in the second chain.

This method, though generally giving fractional solutions, furnishes for $x^3+x^2+1=y^2$ the integral solution (4, 9) and for $x^3+x+1=y^2$ the integral solution (72, 611).

Real Roots of a Class of Reciprocal Equations

By L. S. JOHNSTON, University of Detroit

Consider the reciprocal equations $P(x, s, n)=0$ $(s=1, 2, \cdots; n=1, 2, \cdots)$, separated into two classes,

$$(I)\qquad P(x, s, 2p) = x^{2p} + {}_sH_1x^{2p-1} + {}_sH_2x^{2p-2} + \cdots + {}_sH_{p-1}x^{p+1} + {}_sH_px^p + {}_sH_{p-1}x^{p-1} + \cdots + {}_sH_2x^2 + {}_sH_1 + 1,$$

$$(II)\quad P(x, s, 2p-1) = x^{2p-1} + {}_sH_1x^{2p-2} + {}_sH_2x^{2p-3} + \cdots + {}_sH_{p-1}x^p + {}_sH_{p-1}x^{p-1} + \cdots + {}_sH_2x^2 + {}_sH_1x + 1,$$

where, using the notation of Chrystal,

$${}_sH_r = \frac{s(s+1)(s+2)\cdots(s+r-1)}{r!},\quad {}_sH_0 = 1.$$

We shall need also to use ${}_sC_r$ with its usual meaning in the theory of combinations, and the identities

$$(A)\qquad {}_sH_r = {}_sH_{r-1} + {}_{s-1}H_r,$$

$$(B)\qquad {}_sH_r = {}_{s-1}H_r + {}_{s-1}H_{r-1} + {}_{s-1}H_{r-2} + \cdots + {}_{s-1}H_2 + {}_{s-1}H_1 + 1,$$

$$(C)\qquad {}_sC_r = {}_{s-1}C_r + {}_{s-1}C_{r-1}.$$

We shall prove the following

THEOREM. The equation $P(x, s, n) = 0$ has just k real roots, counting a root of multiplicity j as equivalent to j roots, where $n \equiv k \pmod 4$, and $0 \leqq k < 4$; except that

(a) $P(x, 1, 2p) = 0$ has no real roots,

(b) $P(x, s, 2p-1) = 0$ has just one real root, negative unity, for s less than 3.

PROOF:

(a) $P(x, 1, 2p) = (x^{2p+1}-1)/(x-1)$, and $x^{2p+1}-1 = 0$ has no negative roots.

(b) $P(x, 1, 2p-1) = (x^{2p}-1)/(x-1)$, and $x^{2p}-1 = 0$ has negative unity as its only negative root.

$P(x, 2, 2p-1) = [P(x, 1, p)]\,[P(x, 1, p-1)]$; from the last two statements above it follows that only one of these factors has a real zero, the real zero being at negative unity.

Before attacking the general theorem we remark that a reciprocal equation of odd degree with all signs positive has negative unity as a real root; hence $P(x, s, 2p-1) = 0$ has this root for every s and every p. Furthermore, if a reciprocal equation of even degree has negative unity as a root, this must be a root of even multiplicity. This last statement will be used to show that $P(x, 2, 4r-2) = 0$ has negative unity as a double root for every r.

To prove the general theorem, we shall prove that except for the special cases already proved

(1) $P(x, s, n) = 0$ has at least k real roots, and

(2) $P(x, s, n) = 0$ has no more than k real roots.

For convenience we shall refer to these as Theorem (1) and Theorem (2); we shall also separate $P(x, s, n) = 0$ into four subclasses

(Ia) $P(x, s, 4r) = 0$ (IIa) $P(x, s, 4r-3) = 0$

(Ib) $P(x, s, 4r-2) = 0$ (IIb) $P(x, s, 4r-1) = 0.$

Theorem (1). The theorem is immediately seen to be true for (Ia) and (IIa). For (Ib) we see that $P(0, s, 4r-2) = 1$, and

$$P(-1, s, 4r-2) = -{}_sH_{2r-1} + 2{}_sH_{2r-2} - 2{}_sH_{2r-3} + \cdots + 2{}_sH_2 - 2{}_sH_1 + 2$$
$$= {}_sH_{2r-1} - 2({}_sH_{2r-1} - {}_sH_{2r-2} + {}_sH_{2r-3} \cdots - {}_sH_2 + {}_sH_1 - 1).$$

Using identities (A) and (B) above, this last quantity can be shown to be $-({}_{s-2}H_{2r-1} + {}_{s-2}H_{2r-3} + {}_{s-2}H_{2r-5} + \cdots + {}_{s-2}H_3 + {}_{s-2}H_1)$, which is zero for $s = 2$ and negative for s greater than 2. Hence negative unity is a double root of $P(x, 2, 4r-2) = 0$, and, for s greater than 2, there exist two distinct negative roots of $P(x, s, 4r-2) = 0$.

For (IIb), we remove the factor $(x+1)$ from $P(x, s, 4r-1) = 0$, deriving a new equation

$$\begin{aligned}\overline{P}(x, s, 4r-2) = {} & x^{4r-2} + ({}_sH_1 - 1)x^{4r-3} + ({}_sH_2 - {}_sH_1 + 1)x^{4r-4} + \cdots \\ & + ({}_sH_{2r-2} - {}_sH_{2r-3} + \cdots + {}_sH_2 - {}_sH_1 + 1)x^{2r} \\ & + ({}_sH_{2r-1} - {}_sH_{2r-2} + \cdots - {}_sH_2 + {}_sH_1 - 1)x^{2r-1} \\ & + ({}_sH_{2r-2} - {}_sH_{2r-3} + \cdots + {}_sH_2 - {}_sH_1 + 1)x^{2r-2} \\ & + \cdots \\ & + ({}_sH_2 - {}_sH_1 + 1)x^2 + ({}_sH_1 - 1)x + 1 = 0.\end{aligned}$$

Now $\overline{P}(0, s, 4r-2) = 1$, and $\overline{P}(-1, s, 4r-2)$ can be shown, by using identities (A) and (B) above, to have the value

$$-({}_{s-3}H_{2r-1} + 2{}_{s-3}H_{2r-3} + 3{}_{s-3}H_{2r-5} + \cdots + r{}_{s-3}H_1),$$

which is zero for $s = 3$ and negative for s greater than 3. Hence negative unity is a triple root of $P(x, 3, 4r-1) = 0$, and there exist two distinct negative roots, in addition to negative unity, of $P(x, s, 4r-1) = 0$ for s greater than 3. Theorem (1) is thus completely proved.

To prove Theorem (2) we construct an auxiliary set of equations $Q(x, s, n) = (x-1)^s P(x, s, n) = 0$, and show that $Q(-x, s, n)$ has exactly k variations in sign.

It can be shown that

$$Q(x, s, 2p) = x^{s+2p} - Ax^{s+p-1} + Bx^{s+p-2} - \cdots + (-1)^{s-2}Bx^{p+2}$$
$$+ (-1)^{s-1}Ax^{p+1} + (-1)^s.$$

The terms in x^{s+2p-1}, x^{s+2p-2}, $\cdots$, x^{s+p}, and their corresponding terms in $x, x^2, \cdots, x^p$ vanish, since the coefficient of x^{s+2p-m}, for $m \leqq p$, is

$${}_sH_m - {}_sH_{m-1}{}_sC_1 + {}_sH_{m-2}{}_sC_2 - \cdots + (-1)^{m-1}{}_sH_1{}_sC_{m-1} + (-1)^m{}_sC_m,$$

and this coefficient can be shown, by using identities (A) and (C) in succession, to be zero for every s and every $m \leqq p$. There are, therefore, $s+2p+1-2p=s+1$ terms in $Q(x, s, 2p)$, with s variations in sign; hence the signs alternate throughout the latter polynomial. It is easily shown that $Q(-x, s, 4r)$ has no variations in sign and that $Q(-x, s, 4r-2)$ has just two variations in sign. Hence the general theorem is completely proved for $P(x, s, 2p)=0$.

To prove Theorem (2) for (IIa) we can easily show that

$$Q(x, s, 4r-3) = x^{s+4r-3} - Ax^{s+2r-2} + Bx^{s+2r-3} - \cdots + (-1)^{s-2}Bx^{2r} + (-1)^{s-1}Ax^{2r-1} + (-1)^{s}.$$

The missing terms are explained by the same argument which explains the terms missing from $Q(x, s, 4r-2)$. When s is even, it is easily seen that there is a repeated sign, that of $(-1)^{s/2}$, in the center; when s is odd, the middle term of $Q(x, s, 4r-3)$ vanishes also. $Q(x, s, 4r-3)$ has therefore $s+2$ terms when s is even and $s+1$ terms when s is odd, with s variations in sign in both cases. It can be shown without difficulty that $Q(-x, s, 4r-3)$ has just one variation in sign, which proves the general theorem for (IIa). The proof of Theorem (2) for (IIb) follows exactly the same lines as that for (IIa); we find that $Q(-x, s, 4r-1)$ has exactly three variations in sign. The general theorem is thus completely proved.

It has been shown that negative unity is a triple root of $P(x, 3, 4r-1)=0$. It is also clear that this is the only one of the equations of the general type under consideration which has a triple real root at all, and that none of the other equations of odd degree has even a double real root. It is easily shown that $P(x, 3, 4r-1)=(x+1)^3P(x^2, s, 2r-2)$.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

The Taylor Series. An Introduction to the Theory of Functions of a Complex Variable. By P. Dienes. Oxford University Press, 1931. 552 pages. $7.50.

Professor Dienes has here performed a notable service to the mathematical world in assembling and ordering in a thoroughgoing manner the modern theories relating to the Taylor series. From its title and subtitle one might suppose the book to be another elementary text book on complex function theory from the Weierstrass point of view. The title however is too modest. The book is a pioneer in its field and makes no inconsiderable demands on the maturity of its readers.

The first half of the book, after a foundation consisting of the theory of aggregates and fundamental ideas concerning functions of a real variable, presents most of the material of a first course in complex variable theory with the ideas of Cauchy and Weierstrass uppermost. The seven chapter headings of this part are I, Real Variables; II, Complex Algebra; III, Infinite Series; IV, Elementary Functions; V, Complex Differentiation; VI, Geometrical Language; VII, Complex Integration. The choice of material in these chapters is guided largely by the needs of the second part of the book. The treatment of specific functions (such as the doubly periodic functions) is off the main track of the author's purpose and is therefore dealt with briefly or not at all. On the other hand the geometrical ideas associated with Jordan's theorem, culminating in a treatment of Stieltjes integrals, are carried further than the requirements of a first course. Incidentally the cult of the analyst is shown in the author's use of 'mathematics' and 'geometry' as mutually exclusive terms.

The second half, which is the *raison d'être* of the book, has the following seven chapter headings: VIII, Biuniform Mapping, Picard's Theorem; IX, Representation of Analytic Functions; X, Singularities of Analytic Functions; XI, Overconvergence and Gap Theorems; XII, Divergent Series; XIII, The Taylor Series on its Circle of Convergence; XIV, Divergence and Singularities. The central problem as stated in the author's preface is that of detecting the properties of a function from the sequence of coefficients of its Taylor series and from the formal properties of these series. The pursuance of this central problem leads sometimes to theorems not always associated with Taylor series. Thus the inversion of Taylor series leads up to the classical theorems of Picard (on integral functions and essential singularities) and Riemann (on biuniform mapping). The detection and location of singularities by means of the Taylor coefficients and their relation to the character of the divergence occupies a large share of this half of the book. For this purpose a chapter is devoted to the theory of divergence and summability. This in the reviewer's opinion is not a treatment to which a beginner should come for his first ideas but rather a systematic exposition of the subject in a logical rather than a chronological order.

In regard to the manner of presentation, an effort is made to place due emphasis on important results so that the reader will not lose sight of the forest on account of the trees. This is very necessary in a book with so many laborious proofs which depend on obtaining finer and finer inequalities. Errata, consisting of misprints and slips (only a few of which are in the list of errata at the beginning) are sufficiently numerous to give zest to the reading of the book but will cause no real trouble to the reader. Among these we pause only to mention the curious number of slips in the discussion of arc cos x on page 117 and the example 6 on page 405 in which $g_{k+1}(a)$ should be $\Gamma(kt+t+1)/\Gamma(k+2)$ which necessitates a modification in this proof. In place of footnotes, which would otherwise need to be numerous, a bibliography is placed at the end of the book, to which references throughout the text are made by name and date.

The author makes commendable use of suggestive terms such as *starlike*,

star domain, *analytic landscape* and *picture* (in place of *image*). On the other hand few of us will condone the omission of the word *singular* from the term *essential point*; incidentally the statement on page 232 that "$f(z)$ is completely indeterminate in the neighborhood of an essential point" is misleading. Again *cyclometric* seems a ponderous term for the inverse circular functions and *complex differentiation and integration* for *differentiation and integration of functions of a complex variable* seems to border on slang. There is some confusion between paths and integrals over these paths (pages 247 and 250).

Turning from these minor points let us close by a further reference to the author's main purpose, that of giving a systematic and up to date account of those researches, mostly by European mathematicians, into the properties of functions of a complex variable in so far as they relate to the expression of these functions by power series. The author's own researches form an important contribution to the subject. The book is a difficult and useful piece of work well done and forms a significant addition to mathematical literature.

NORMAN MILLER

Catalogue of an Exhibition at Columbia University to Commemorate the One Hundredth Anniversary of the Birth of Lewis Carroll (Charles Lutwidge Dodgson) 1832–1898. Columbia University Press, New York, 1932. 153 pages $.35.

Lewis Carroll would have been the last man to refer to himself as a mathematician. He always insisted that Lewis Carroll was the author of the "Alice" books and that Charles Lutwidge Dodgson was the author of such mathematical works as *Condensation of Determinants*. Since we know that the familiar name is a variant of the unfamiliar one through "Lewis = Ludovicus = Lutwidge; Carroll = Carolus = Charles," we shall speak of him by either name. The world of his day learned with much astonishment of the identity of these two men. The world of the present moment will not give the same start of surprise over it.

No claim can be made for Dodgson as an original mathematician. He did little or no research work and was the originator of no epoch-making ideas in his field. His publications in the realm of mathematics relate rather to improvements and aids for the parts of the science which he presented in his university instruction. The Exhibit includes (1) the first edition of *Notes on the First Two Books of Euclid*, 1860, his first publication; (2) the first edition and "the author's own copy, with his initials 'C.L.D.' on front end paper, and corrections in pen" of *A Syllabus of Plane Algebraical Geometry*, 1860, his first book; (3) the first edition of *The Formulae of Plane Trigonometry*, 1861; (4) *Open Letter to Mathematical Teachers*, 1862; (5) *A Guide to the Mathematical Student*, 1864; (6) *Condensation of Determinants*, 1866; (7) *An Elementary Treatise on Determinants*, 1867; (8) *The Fifth Book of Euclid Treated Algebraically*, 1868; (9) *Euclid, Book V Proved Algebraically*, 1874; (10) *Examples in Arithmetic*, 1874; (11) *Euclid Books I, II*, 1875; (12) *Euclid and His Modern Rivals*, 1879; (13) *A Tangled Tale* (Answers to Knots I and II), 1880; (14) *A Tangled Tale* (Knots I to VII), 1880–1882, "*A Tangled Tale* is a series of mathematical problems written as sto-

ries in the usual Carrollian style"; (15) *Euclid Books I, II*, 1882; (16) *Lawn Tennis Tournaments*, 1883; (17) *The Principles of Parliamentary Representation*, 1884, "A mathematical treatment of the English method of voting"; (18) *Divisibility by Seven*, 1885; *Note on Question 7695*, 1885; (19) *Curiosa Mathematica*. Part I *A New Theory of Parallels*, 1888, Part II *Pillow Problems Thought out during Sleepless Nights*, 1893; (20) *Brief Method of Dividing a Number by 9 or 11*, 1897; all of them in first editions. These are all of his important works relating to mathematics. The Catalogue lists six "Principal Works" and three of these are numbers (2), (12) above and *Symbolic Logic*.

The "Alice" material is the most conspicuous in the Exhibit and in the Catalogue, as is natural, and yet all the items seem to be related to one another. Carolyn Wells in one of the many brief newspaper articles appearing at this time put it thus: "But be it remembered, too, logic and nonsense are not mutually exclusive." Indeed, there is hardly a case which fails to reveal Dodgson's mathematical tendencies. It may be a list of persons whom he wishes to keep in distinct groups and so distinguishes the groups by $+$, $\times$, $\surd$. It may be a parody such as "Hiawatha's Photographing" in which the last lines of the first stanza read:

> "Till it looked all squares and oblongs,
> Like a complicated figure
> In the Second Book of Euclid."

It may be a word-puzzle called a "Syzygy" and illustrated by the two words "door" and "window" with the following chain linking them together: "door (oor), poorest (res), resound (und), undo (ndo), window."

The Exhibition contains four hundred and fifteen items and no less distinguished an authority than President Butler of Columbia University is responsible for the statement that "it is the most extraordinary exhibit that ever has been or could be gathered together of Lewis Carroll." Many of the items as listed in the Catalogue are accompanied by explanatory notes and so a wealth of information is provided for future reference. The items are classified under: "The 'Alice' Books," with the outstanding "Original Manuscript of *Alice's Adventures Under Ground*" and the table on which it was supposedly written; "Translations of the 'Alice' Books," with first editions of most of the translations into thirteen languages; "Miscellaneous Works"; "Mathematical Works"; "Works on Logic"; "Pamphlets and Leaflets"; "Games and Puzzles"; and others containing material based on Carroll's work. The Catalogue is illustrated to the extent that it contains a reproduction of a miniature of Lewis Carroll, a photograph of "Alice" taken by Carroll, and also a photograph of Carroll with his camera lens.

The Catalogue, the Exhibit, the Centenary Celebration all combined to make a memorable and worthy tribute to Lewis Carroll. To many the personal contact with "Alice" afforded by her appearance at the Celebration was an untold satisfaction. One who stood near her when the degree was conferred upon

her heard her say that she did not stand there alone but that the spirit of Lewis Carroll accompanied her and that she received the honor in his stead.

LAO G. SIMONS

Outline of the History of Mathematics. By Raymond Clare Archibald. The Society for the Promotion of Engineering Education. No. 18. February, 1932. 53 pages $.30.

This work appears as "Bulletin Number 18 of the Summer School for Engineering Teachers," an indication of the end in view in its preparation.

The "Outline" is far from corresponding to the usual meaning of that term. It is that, to be sure, but it is far more. It is an interesting account of the history of mathematics with emphasis on only the important persons and contributions to the field. Here and there less important items are introduced to enliven the lecture but these always flow naturally from the matter which they follow. We have, for example, the connection of Gauss with Dase: (p. 44) "Gauss was a notable instance of a youthful arithmetical prodigy who later achieved greatness in mathematics. The most extraordinary mental calculating prodigy who ever lived was Zacharias Dase of Hamburg who died in 1861, aged 37. He multiplied two 8-figure numbers in 54 seconds, two 20-figure numbers in 6 minutes, two 40 figure numbers in 40 minutes and two 100-figure numbers in $8\frac{3}{4}$ hours, and found the square root of a hundred figure number in 52 minutes. Gauss tried to turn his abilities to some useful purpose and as a result we have Dase's seven-place tables of the natural logarithms of numbers from 1 to 10,500 (1850) and his factor tables of all numbers in the sixth, seventh, and eighth million (1862–65)." One other of quite a different kind may be included: (p. 28) "Purely as a thrilling intellectual experience, without any imaginable practical application, Apollonius of Perga, and other Greeks, developed a marvelous body of knowledge with regard to conic sections. Then suddenly, 1800 years later, to a Kepler this knowledge had most illuminating practical applications."

There is a touch of genius in saying familiar things in a new way. An author who does this adds something to his subject at the same time that he is presenting its various phases. This author in his "Outline" makes a contribution in more ways than the name indicates.

The "Literature List" which closes the book is a fine set of references to books and magazine articles bearing on the contents of the work. These references are referred to by number as the various mathematicians and mathematical concepts are introduced. They present a wide field for further study in each direction. The list should be used by libraries to check up their resources bearing on the history of mathematics.

Table of Contents. Prefatory Note. Synopsis. History of Mathematics before the Seventeenth Century: A. Babylonian and Egyptian Mathematics 3500 B.C.–600 B.C., B. Greek Mathematics 600 B.C.–500 A.D., C. Hindu, Arabic, Persian Mathematics 500–1200, D. European Mathematics 1200–1600. History of

Mathematics after the Sixteenth Century: A. The Seventeenth Century, B. The Eighteenth Century, C. The Nineteenth Century and Later.

To an instructor in the history of mathematics this presentation will offer a variation in the handling of the course, and will be most welcome for that reason. An index would have increased its usefulness here. It will be welcome as well for the very readable story told so well and so authoritatively, which latter reason recommends it to any reader.

LAO G. SIMONS

College Algebra. By Louis J. Rouse. New York, John Wiley & Sons, 1931. x+345 pages. $2.25.

The preface to this book states that "This book, though primarily intended for use as a text-book in the freshman year of colleges and technical schools, is suitable for use in any school offering a course in advanced algebra." This statement, that the book may be used in schools of lower grade than the college, seems to the reviewer to be necessary, for otherwise the amount of elementary material included seems unnecessarily large for a college text. To be sure, it is claimed that this can serve as a basis of review for the college student, but it seems as if he should be able to do this reviewing by himself, with the aid of his previous text-books. The only advantage from the point of view of review is that the material is easily accessible for reference. However, the inclusion of a large amount of elementary review material is not peculiar to this text, but seems to be a feature of many of the present day college algebras.

In a few places the book is not so inclusive as it might be: for example, in the chapter on the theory of equations, no treatment is given of zero and infinite roots. The explanations and developments of theory are well presented in considerable detail; and we found that when the book was actually tried with a class, the students grasped these explanations more readily than they do the more concise proofs given in some other frequently used texts.

There is an abundance of well chosen problems, so that the student is afforded ample practice. Those on determinants, mathematical induction and the binomial theorem are especially good. In this connection, we observe that the authors state that "the many solved examples are intended not so much to serve as models for working similar exercises, as to illustrate the principles and new theories developed in the text. The rules given are usually to be considered as statements of processes which have been previously discussed rather than to be used mechanically in solving problems." This is a worthy aim in itself, but we fear that it will not be realised, and that the average student will do exactly what the author does not want him to do: he will use the rules and solved examples as short cuts to aid in solving other problems, without having to spend much time studying the underlying theory.

In general, we may say that the book contains the usual material, well-presented, and with ample problems. There is nothing extraordinary in either the choice or the presentation of material, nor any such originality of treatment

as in the case of the recent algebra by Smail. But the book, though not unusual, is a good one, and may be recommended for use.

Before concluding, we wish to call attention to a minor point of criticism: the development of a three-rowed determinant by an elaborate scheme of curved lines, as given on page 221, seems an unnecessary complication. It is difficult for the student to learn the exact positions of the lines, and they are easily forgotten. In view of the fact that the student will sooner or later have to learn the development by minors, it seems more logical to develop three-rowed determinants in this way from the start. The student "catches on" quickly, and in the reviewer's experience always remembers it.

MILDRED WATERS DEAN

Ordinary Differential Equations. By E. L. Ince. London, Longmans, Green and Co., 1927. vi+558 pages.

This volume, "the first to be launched into the world by a member of the staff of the newly-founded Egyptian University," seems to have received curiously little attention from reviewers. It is a very useful book for study or reference, treating as it does all phases of its subject.

Part I deals with the real domain. Introductory chapters on formal solution are followed by chapters on existence theorems, the Lie theory, the properties of linear equations in general and those with constant coefficients in particular, solutions by infinite series and by definite integrals, and the modern forms of the Sturm theories. Part II deals with the complex domain. After proving the usual existence theorems, the author proceeds first to non-linear equations; then discusses linear equations, including solutions by series and by contour integration, and the study of linear systems; the part closes with a chapter on oscillation theorems in the complex domain. Appendices deal with the history of formal integration, with numerical integration, and with bibliography.

The form of treatment is not quite uniform. The early chapters on formal integration are written entirely in the spirit of the usual elementary text-book, with the familiar vagueness as to what a *solution* of a differential equation really is, and with no critical examination of the accuracy of the steps used in "solving" the equation. The chapter on continuous groups exhibits the optimism regarding the behavior of the functions, which is expected of writers in this field. In practically all the rest of the book, theorems are precisely stated and proved with scrupulous attention to logical detail. It seems likely that the author has followed his sources rather closely, both as to matter and manner. For instance, the chapters on algebraic properties, Sturmian theorems, and boundary value problems are even in small detail like the original papers of Birkhoff and Bôcher.

It is very convenient to have so excellent a treatment of so many phases of the theory of ordinary differential equations gathered into one volume, and every worker in this branch of analysis must be grateful to the author for the valuable service rendered by his book.

The typography is excellent; errors are few and trivial.

W. A. HURWITZ

Note on a Greek Papyrus in Vienna

There has recently been published by the Nationalbibliothek at Vienna, in its *Mitteilungen aus der Papyrussammlung*, "Eine stereometrische Aufgabensammlung im Papyrus Graeccus Vindobonensis 19996," an event of considerable interest in the history of mathematics. The translation and extended description is the work of Dr. H. Gerstinger of Vienna, who read and transcribed the text; Dr. Oellacher of Salzburg; and Dr. K. Vogel, who prepared the principal part of the text,—"Kapital C. Mathematischer Inhalt des Papyrus." Dr. Vogel has already proved himself to be one of the coming historians of mathematics, and his present contribution confirms the impression which his previous articles have made.

The manuscript consists of a number of fragments of a papyrus roll containing thirty-eight problems relating to the metrical side of solid geometry, twenty-three being accompanied by drawings. Although the manuscript is not unique as to content, since it is quite like two others that have been described, one in the Field Museum in Chicago and the other in Berlin, it loses nothing of its value because of this fact.

The Greek text is given so far as it is decipherable, after which Dr. Vogel discusses at length the nature and significance of the work. He first gives a general survey of the manuscript, followed by a detailed statement of the methods employed by the writer and his Greek predecessors. In the discussion he pays special attention to the truncated pyramid which has recently been the subject of so much investigation in connection with the Moscow Papyrus (*c.*1850 B.C.), and upon which he has already taken a definite stand in his article in the *Journal of Egyptian Archaeology* (vol. 16, 1830, pp. 242–249). He then gives a list of mathematical terms used in the papyrus, together with abbreviations, symbols, and certain rather unusual number forms, both integral and fractional. The list will be particularly helpful to students of Greek mathematics as giving certain variants of the common numerals that were used by such writers as Diophantus and Nicomachus.

Dr. Vogel then gives a translation of the entire text, part of which is necessarily conjectural, and adds numerous explanatory notes. This part of the article (pp. 54–72) will naturally be considered the most important. It includes the mensuration of the rectangular solid, the pyramid, frustum of a pyramid, prism, cylinder, cone, and truncated cone, but not the sphere.

The manuscript seems to have been a schoolbook; that is, a copy made by a pupil. It was found in Dimêh, the ancient Soknopaiu Nesos, in the Fayum. The date is uncertain, but Dr. Vogel places it before the fourth century to which Professor R. C. Archibald had tentatively assigned it (Chace ed. of the Rhind Papyrus, vol. I, p. 124).

The manuscript was evidently written early in the Christian era and the date will doubtless be found with a satisfactory approach to accuracy after the work has been subjected to further study.

David Eugene Smith

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D. C.

All reports of club activities, suggestions and topics for club programs, and material of interest to clubs should be sent to F. M. Weida, The George Washington University, Washington, D. C. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

CLUB ACTIVITIES

A.

THE PI MU EPSILON MATHEMATICAL FRATERNITY

Pi Mu Epsilon is an academic fraternity in institutions of university grade. Its primary aim is the advancement of mathematics and scholarship. It is a living, active, working fraternity of scholars in which the members are actively engaged in study and research in the preparation of papers in the field of mathematical science to be presented at its regular meetings.

CHAPTER REPORTS

1931–1932

Alabama Alpha Chapter of Pi Mu Epsilon.

The officers for 1931–1932 were: Joseph S. Gelders, Director; Eric Rodgers, Vice Director; William F. Adams, Secretary; H. S. Thurston, Treasurer; Sara Ella Haughton, Librarian.

Alabama Alpha began the year with twenty-two active members. In October 27, 1931, thirteen new members were initiated. On March 29, 1932, four new members were initiated. This brings our total up to thirty-nine active members.

The meetings and programs were as follows:

September 29, 1931: "A problem in collineation groups" by Mr. Fred A. Lewis.

November 24, 1931: "Euclid's parallel postulate" by Miss Sara Ella Haughton.

February 2, 1932: "The use of mathematics in economics" by Mr. Paul Kringer.

February 23, 1932: "Extension of differentiation and integration" by Mr. G. N. Carmichael.

March 29, 1932: "A complete system of differential equations" by Mrs. Charles E. Watkins.

April 26, 1932: "Some geometry in connection with the bilinear transformation" by Mrs. J. W. Sledge; "A study of the second degree equation in isotropic coordinates" by Miss Margaret Gorrie; "Analytical proofs of some theorems on the triangle" by Miss Boyce Garrett; "A study of seismology" by Mr. Eric Rodgers.

Alabama Alpha has held two social meetings—a bridge party in the fall and a picnic this spring.

In addition to the above activities, we have contributed fifty dollars to the storm victims in this locality.

WILLIAM F. ADAMS, *Secretary*

Arkansas Alpha Chapter of Pi Mu Epsilon.

The officers for 1931–1932, elected May, 1931, by vote of the Chapter were: Guilford V. Smith, Director; Roberta Currie, Vice Director; Marion Brashiers, Secretary; Martha Bond, Treasurer; Lulu Mae Holland, Librarian; Mr. Paul Cramer, Faculty Advisor.

We now have twenty-eight active members. Seven new members were initiated at a banquet March 31, 1932. Our meetings were held monthly, and our programs were as follows:

October 21, 1931: "Mathematics as the tool of science" by Professor Roberds.

November 19, 1931: "History of the trisection of an angle" by Robert Vining; "Analysis situs" by Dr. V. W. Adkisson.

February 10, 1932: "Lewis Carroll" by Dr. H. M. Hosford; "The golden section" by Guilford Smith.

March 15, 1932: "Arithmetic of 100 years ago" by Miss Helen Graham.

March 31, 1932: "Contribution of science to 'Believe it or not' " by Frank Davis; "History of Pi Mu Epsilon" by A. B. Carson.

April 14, 1932: "Exponential form of complex numbers" by L. M. Martin; "Mathematics in correlation charts" by Dr. J. R. Guberich.

May 17, 1932: "Mathematicians of Michigan" by Jane Stelzner; "Mathematicians of California" by Louise Love.

A picnic was held on the campus for pledge services March 22, 1932, and the activities of the year ended with a picnic at Ghost Hollow on Saturday afternoon, April 30, 1932.

MARION J. BRASHIERS, *Secretary*

B.

LOCAL MATHEMATICS CLUBS

The Mathematics Klub of Adelphi College.

The officers for 1931–1932 were: Dr. Joseph Bowden, Honorary President; Theresa Cartereau, President; Sarah Gordon, Vice President; Dorothy Hill, Secretary; Dorothy Hagemann, Treasurer.

The officers are elected in May of each year by vote of the Klub. There were 32 active members this year. The purpose of the Klub is to stimulate further interest in mathematics outside of regular college courses. Membership is open to any student interested in mathematics. Regular meetings are held twice a month, On November 17, Dr. Dalman, professor of Chemistry at Adelphi discussed at the meeting the importance of Mathematics in the Development of Science. Our president, Miss Cartereau, read an article to the Klub "What is New in Math" on December 3. At the next meeting, December 5, one of the members, Miss Adele Shrage, spoke on "Oddities in Mathematics." January 5, our president gave a report on "Place-Value in the Number System." This was followed, February 20, by "Geometric Fallacies" reported on by Miss Madeline Sniffen. April 11, Miss Sarah Gordon, Vice-President of the Klub, discussed Newton, the Calculus and Leibnitz. During the year besides the regular meetings, a party and two supper meetings were held. The party was given to welcome new Freshmen on October twentieth. Bridge was played and refreshments served. The two supper meetings were December third and May third, the first at the college dining room, and the second at a restaurant in Hempstead, Long Island. Alumni were invited and several attended during the year. The regular meetings ended this year with the nominations of officers for 1932–1933.

DOROTHY HILL, *Secretary*

The Mathematics Club of the Cooper Union Institute of Technology.

The officers for 1931–1932 were: G. Kosolapoff, '32, President; W. W. Rigrod, '33, Vice President; C. H. Kropp, '33, Secretary; E. H. Ryan, '33, Treasurer; J. Watkins, '33, Assistant Secretary.

The club has a membership of 216.

The meetings and programs were as follows:

November 9, 1931. "Generalization in mathematics" by Prof. T. S. Fiske, Columbia University.

November 24, 1931. "Higher plane curves" by J. F. Skelly, '34.

December 15, 1931. "Theory and use of the slide rule" by E. A. Drago, '32.

January 12, 1932. "Non-Euclidean geometry" by A. Rubinsky, '34.

January 26, 1932. "The mathematics of lens surfaces" by A. Ginsberg, '32.

February 9, 1932. "Systems of coordinates" by K. Itkin, '34.

February 23, 1932. "Elementary vector analysis in physics" by D. Briansky, '32.

March 8, 1932. "Transformation of a theorem in geometry" by L. Green, '34.

March 22, 1932. "Determinants" by C. A. Wamser, '34.
April 5, 1932. "Matrices" by Mr. C. H. Lehmann, Instructor, Department of Mathematics; Election of Officers.

C. H. LEHMANN, *Faculty Adviser*

The Mathematics Club of the University of Buffalo.

The Mathematics Club of the University of Buffalo, organized in 1929, is open to all students of mathematics. At present there are about twenty members. The club holds monthly or bi-monthly meetings, when students or outside speakers give the program. The officers for the year 1932 were: William Corse, President; Alice Link, Vice-president; Oakland Becker, Secretary-treasurer.

The Mathematics Club is very proud to have established this year the Wilfred H. Sherk Memorial Prize in Mathematics. This is an annual prize to be awarded for the best student paper in mathematics, and was established in memory of the late Professor Sherk who was for many years the head of the department of mathematics at the University of Buffalo. The recipient of the award for the year 1932 is John W. Wrench of the class of 1933 whose subject was "The impossibility of solving the quintic."

The programs of the year follow:
November: "Mathematics and psychology" by Benjamin B. Sharpe; "Number systems" by Robert R. Lyle.
February: "Changing methods in teaching secondary school mathematics" by Miss Mary Crofts of Fosdick-Masten Park High School.
April: April fool puzzle party.
May: The club was entertained at the home of Professor Gehman. The Sherk Prize was awarded and plans for the coming year discussed.

The following is the basis for determining the winner of the Wilfred H. Sherk Memorial Prize in Mathematics:

1. This prize of five dollars shall be awarded once a year to the student who has submitted the best paper on any branch of mathematics, pure or applied.
2. All undergraduate students of mathematics in the University of Buffalo are eligible to compete.
3. The paper need not be entirely original, but must be the product of independent research by the student. It must be accompanied by an adequate bibliography.
4. The paper must be submitted to the judges on or before April 15 of each year.
5. The name of the winner will be announced at the May meeting of the Mathematics Club.
6. The judges shall consist of two students, chosen by vote of the Mathematics Club, and three members of the mathematics department.
7. The award shall be given to one who receives the majority vote of the board of judges.
8. The judges shall base their decision on the following points: (a) Evidences of careful independent research; (b) Originality; (c) Literary style and form of presentation; (d) Mathematical background of the student. This should make it possible for freshmen and sophomore papers to be judged on a par with junior and senior papers.

OAKLAND BECKER, *Secretary-Treasurer*

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing, and with margins at least one inch wide.

PROBLEMS FOR SOLUTION

N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

3562. *Proposed by Vladimir F. Ivanoff, San Francisco, Calif.*

If $\phi(x, y, z)=0$, $\psi(x, y, z)=0$, prove that

$$\begin{vmatrix} d^2x & d^2y & d^2z \\ \phi_x & \phi_y & \phi_z \\ \psi_x & \psi_y & \psi_z \end{vmatrix} = 0,$$

under suitable conditions.

3563. *Proposed by Otto Dunkel, Washington University.*

To construct approximately an angle which is $1/n$th of a given angle, let the given angle be BAC, with C as any convenient point on the side AC. Take D on BA produced so that $AD=AC$, and draw DC. Lay off from A along BA produced lengths equal to $(n-1)DC/2$ and $(n-1)AC$ terminating, respectively, in M and N. Determine the point P which divides MN in the ratio $(8-n)$: $4(n-2)$; then the angle BPC is approximately $1/n$th of the angle BAC.

This construction is exact for $n=2$, 4 and for the trivial case $n=1$. For $n=3$ it is essentially the method given by Wedderburn in the second part of his problem 2972 [1922, 224]; and the error in this case was discussed in the solution [1925, 96–97]. For $n=3$ a more convenient but less accurate method may be obtained by taking P as the mid-point of MN. Determine the approximate error for given angles not greater than 45° for the cases $n=5$, 7, and for the modified construction when $n=3$.

3564. *Proposed by H. T. R. Aude, Colgate University.*

A determinant of order $n+1$ has for the elements of its first and second rows, respectively, the successive powers of α and x with exponents from 0 to n inclusive. The third row is the derivative of the second row, and the elements of any row after the second are given by the relation

$$\alpha_{i+1,j} = \frac{1}{i-1}\frac{d\alpha_{i,j}}{dx}, \quad i = 2, 3, \cdots, n.$$

Prove that this determinant has the value $(x-\alpha)^n$.

3565. *Proposed by Orrin Frink, Jr., Pennsylvania State College.*

Find the ellipse of least area circumscribing a given triangle.

SOLUTIONS

3500 [1931, 340]. *Proposed by Emma M. Gibson, Springfield High School, Springfield, Mo.*

Show that the primitive of the differential equation, $p^2(1-x^2)=(1-y^2)$ is $x^2+y^2-2Axy=1-A^2$ and derive this equation by taking the sine of the sum of two angles, both of which are arcsines.

Solution by J. D. Leith, University of North Dakota.

Extracting the square root of each side of the differential equation, separating the variables, and integrating, we obtain the primitive at once in the form $\arcsin y=\arcsin x+\arcsin k$, where the plus sign alone has been used on the right. Taking the sine of each side, and setting $k^2=1-A^2$, we obtain the required result after isolating on one side the radical term containing x and then squaring each side. Since A may be positive or negative, the double sign has been taken into account in the result.

Also solved by H. M. Feldman, F. Underwood, and the proposer.

3503. (1931, 408) *Proposed by W. H. Rasche, Virginia Polytechnic Institute.*

The apex angles of two cones of revolution are α_1 and α_2; the cones intersect in an ellipse whose semi-axes are a and b; show that the angle β which the axes of the cones make with each other is given by the equation

$$\cos\beta = [\sin^2\tfrac{1}{2}\alpha_1 - (b/a)^2\cos^2\tfrac{1}{2}\alpha_1]^{1/2}[\sin^2\tfrac{1}{2}\alpha_2 - (b/a)^2\cos^2\tfrac{1}{2}\alpha_2]^{1/2} - [1-(b/a)^2]\cos\tfrac{1}{2}\alpha_1\cos\tfrac{1}{2}\alpha_2.$$

Solution by F. Underwood, University College, England.

Let γ_1 and γ_2 be the angles which the axes of the cones make with the plane of the ellipse. (See the solution, immediately following, of problem 3504 for the relation $\cos\gamma_1=e\cos\tfrac{1}{2}\alpha_1$.) The angle β between the axes of the two cones may be $\gamma_1-\gamma_2$, $\gamma_1+\gamma_2$, or $\pi-\gamma_1-\gamma_2$ according to the senses in which the angles γ_1 and γ_2 are measured, and according as an acute or obtuse angle is taken as β. Now

$$\cos\gamma_1\cos\gamma_2 = e^2\cos\tfrac{1}{2}\alpha_1\cos\tfrac{1}{2}\alpha_2 = \{1-(b/a)^2\}\cos\tfrac{1}{2}\alpha_1\cos\tfrac{1}{2}\alpha_2 = p, \text{ say.}$$

If γ_2 is acute (as well as γ_1),

$$\begin{aligned}\sin\gamma_1\sin\gamma_2 &= (1-e^2\cos^2\tfrac{1}{2}\alpha_1)^{1/2}(1-e^2\cos^2\tfrac{1}{2}\alpha_2)^{1/2}\\ &= \{\sin^2\tfrac{1}{2}\alpha_1 + (b/a)^2\cos^2\tfrac{1}{2}\alpha_1\}^{1/2}\{\sin^2\tfrac{1}{2}\alpha_2 + (b/a)^2\cos^2\tfrac{1}{2}\alpha_2\}^{1/2}\\ &= q, \text{ say.}\end{aligned}$$

Then

(a) if $\beta = \gamma_1 - \gamma_2$, $\cos\beta = p + q$

(b) if $\beta = \gamma_1 + \gamma_2$, $\cos\beta = p - q$

(c) if $\beta = \pi - \gamma_1 - \gamma_2$, $\cos\beta = -p + q$

Also solved by W. B. Campbell, A. Pelletier, and Paul Wernicke.

3504 (1931, 408). *Proposed by W. H. Rasche, Virginia Polytechnic Institute.*

Two cones of revolution whose apex angles are α_1 and α_2 intersect in an ellipse whose semi-major and semi-minor axes are a and b; show that

$$\cos\gamma_1/\cos\gamma_2 = \cos\tfrac{1}{2}\alpha_1/\cos\tfrac{1}{2}\alpha_2$$

in which γ_1 and γ_2 are the angles which the axes of the cones make with the plane of the ellipse.

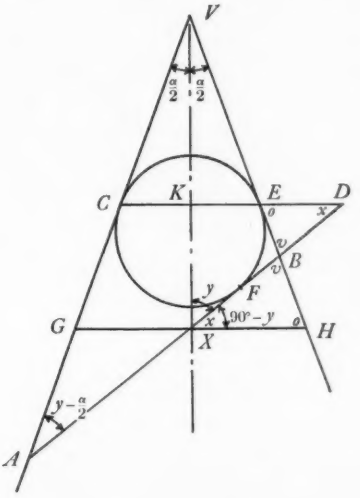
Solution by V. F. Murray, Hoboken, N.J.

Let a circular cone of vertical angle $AVB = \alpha$ be cut by a plane AXB making an angle $VXB = \gamma$ with the axis VX (see figure).

Inscribe the sphere FCE touching the plane AXB in the point F and the cone in the circle CKE. Let the planes AXB, CKE meet in the line D. Draw the line XH parallel to CE, to meet VE in H.

The point F is a focus of the ellipse and the line D is the corresponding directrix (see C. Smith, Geometrical Conics, 1894, p. 180).

The eccentricity e equals BF/BD or BE/BD, BF and BE being equal tangents.



In the triangle BXH,

$$\begin{aligned}\sin BXH/\sin BHX &= BH/BX \\ &= BE/BD, \text{ by similar triangles,} \\ &= e.\end{aligned}$$

Since VXH is a right angle, $\sin BXH = \cos\gamma$ and $\sin BHX = \cos\frac{1}{2}\alpha$. Thus $\cos\gamma/\cos\frac{1}{2}\alpha = e$.

The construction given is quite general so that when a circular cone of apex angle α is cut by a plane making an angle γ with the axis, the eccentricity of the resulting conic section is given by $\cos\gamma/\cos\frac{1}{2}\alpha$.

In the problem, the ellipses being given the same, the eccentricities obtained from the two cones are the same, therefore

$$\cos\gamma_1/\cos\tfrac{1}{2}\alpha_1 = \cos\gamma_2/\cos\tfrac{1}{2}\alpha_2$$

or

$$\cos\gamma_1/\cos\gamma_2 = \cos\tfrac{1}{2}\alpha_1/\cos\tfrac{1}{2}\alpha_2.$$

This is true for all parallel planes in each case. To determine VX corresponding to a major axis $2a$; in the triangle VAB,

$$VB = \frac{2a\sin(\gamma - \frac{1}{2}\alpha)}{\sin\alpha},$$

and in the triangle VBX,

$$VX = VB\,\frac{\sin(\gamma + \frac{1}{2}\alpha)}{\sin\gamma} = 2a\,\frac{\sin(\gamma + \frac{1}{2}\alpha)\sin(\gamma - \frac{1}{2}\alpha)}{\sin\alpha\sin\gamma}.$$

from which the values of VX in the two cones can be calculated.

Also solved by W. B. Campbell, J. B. Ennis, A. Pelletier, F. Underwood, and Paul Wernicke.

3506 [1931, 408]. *Proposed by E. B. Escott, Oak Park, Illinois.*

Solve these n simultaneous equations in n unknowns:

$$\begin{vmatrix} x_1x_2\cdots x_{n-1} \\ x_nx_1\cdots x_{n-2} \\ \cdots\cdots\cdots \\ x_3x_4\cdots x_1 \end{vmatrix} = a_1; \quad \begin{vmatrix} x_2x_3\cdots x_n \\ x_1x_2\cdots x_{n-1} \\ \cdots\cdots\cdots \\ x_4x_5\cdots x_2 \end{vmatrix} = a_2; \cdots$$

$$\begin{vmatrix} x_nx_1\cdots x_{n-2} \\ x_{n-1}x_n\cdots x_{n-3} \\ \cdots\cdots\cdots \\ x_2x_3\cdots x_n \end{vmatrix} = a_n.$$

Solution by T. L. Smith, Carnegie Institute of Technology

Let $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$ be a set of n mutually orthogonal unit vectors along the corresponding coordinate axes in n-space. Then the system of equations given above may be more compactly written in the form

$$\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \cdots \mathbf{e}_n \\ x_n & x_1 & x_2 \cdots x_{n-1} \\ x_{n-1} & x_n & x_1 \cdots x_{n-2} \\ \cdots & \cdots & \cdots \\ x_2 & x_3 & x_4 \cdots x_1 \end{vmatrix} = a_1\mathbf{e}_1 \pm a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \pm a_4\mathbf{e}_4 + \cdots,$$

where the double sign is $+$ or $-$ according as n is odd or even.

From this form, the vector with components $(a_1, \pm a_2, a_3, \cdots)$ is perpendicular to each of the $n-1$ vectors whose components are the elements in the rows of the determinant in the last equation. This fact gives rise to the $n-1$ equations

$$\begin{cases} a_1x_n \pm a_2x_1 + a_3x_2 \pm \cdots = 0, \\ a_1x_{n-1} \pm a_2x_n + a_3x_1 \pm \cdots = 0, \\ a_1x_{n-2} \pm a_2x_{n-1} + a_3x_n \pm \cdots = 0, \\ \cdots\cdots\cdots\cdots\cdots, \end{cases}$$

which on rearrangement becomes

$$\begin{aligned} \pm a_2x_1 + a_3x_2 \pm a_4x_3 + \cdots &= 0, \\ a_3x_1 \pm a_1x_2 + a_5x_3 \pm \cdots &= 0, \\ \pm a_4x_1 + a_5x_2 \pm a_6x_3 + \cdots &= 0, \\ \cdots\cdots\cdots\cdots\cdots &. \end{aligned}$$

The solution of this system of $n-1$ homogeneous linear equations determines the ratio of the unknowns,

$$x_1 : x_2 : \cdots : x_n = \alpha_1 : \alpha_2 : \cdots : \alpha_n,$$

where

$$\alpha_1 = \begin{vmatrix} a_3 \pm a_4 \cdots \\ \pm a_4 \quad a_5 \cdots \\ \cdots\cdots\cdots \end{vmatrix}, \quad \alpha_2 = \begin{vmatrix} \pm a_4 \quad a_5 \cdots \\ a_5 \pm a_6 \cdots \\ \cdots\cdots\cdots \end{vmatrix}, \quad \cdots, \quad \alpha_n = \begin{vmatrix} \pm a_2 \quad a_3 \cdots \\ a_3 \pm a_4 \cdots \\ \cdots\cdots\cdots \end{vmatrix}.$$

($+$ or $-$ according as n is odd or even).

Finally, we may solve for any of the unknowns, say x_r, by eliminating all the other unknowns from one of the original given equations, say the first, by means of the relations

$$x_i = \alpha_i x_r / \alpha_r.$$

On substituting these relations in the first of the given equations, multiplying both members by α_r^{n-1} for symmetry, and factoring out x_r^{n-1}, we obtain

$$x_r^{n-1}\begin{vmatrix} \alpha_1 & \alpha_2 \cdots \alpha_{n-1} \\ \alpha_n & \alpha_1 \cdots \alpha_{n-2} \\ \cdots & \cdots \\ \alpha_3 & \alpha_4 \cdots \alpha_1 \end{vmatrix} = a_1\alpha_r^{n-1}.$$

Whence finally

$$x_r = \alpha_r a_1^{\frac{1}{n-1}}\begin{vmatrix} \alpha_1 & \alpha_2 \cdots \\ \alpha_n & \alpha_1 \cdots \\ \cdots & \cdots \\ \alpha_3 & \alpha_4 \cdots \end{vmatrix}^{\frac{1}{n-1}}.$$

In this final result, the radical may be interpreted as any one of the $(n-1)$th roots of the quantity underneath it. The determinant under the radical is obtained from the determinant in the first of the given equations by replacing the x_i by the corresponding α_i.

Also solved by W. V. Parker and F. Underwood.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

Professor R. A. Millikan, of the California Institute of Technology, has been awarded the Roosevelt medal of the Roosevelt Memorial Association.

The Franklin Institute has awarded a Cresson medal to Professor P. W. Bridgman, of Harvard University.

Juan de la Cierva has been awarded the Guggenheim gold medal for the promotion of aeronautics, for his work in the development of the theory and practice of the autogyro.

Mrs. Anna Johnson Pell-Wheeler of Princeton, N. J. received the honorary degree of Doctor of Science at the eleventh annual commencement exercises of New Jersey College of Women, June 4, 1932.

Professor Marston Morse, of Harvard University, has been elected a member of the National Academy of Sciences.

The following have been elected members of the American Academy of Arts and Sciences: C. R. Adams, A. A. Bennett, Einar Hille, and D. V. Widder.

Professor Werner Heisenberg, of the University of Leipzig, is taking part in the fifth annual symposium in theoretical physics at the University of Michigan, June 27–August 19, 1932.

Dr. Beulah M. Armstrong, of the University of Illinois, has been promoted to be associate in mathematics.

Dr. H. W. Bailey has been promoted to an assistant professorship at the University of Illinois.

Assistant Professor B. H. Brown has been promoted to a professorship of mathematics at Dartmouth College.

J. P. DenHartog, chief of the dynamic section of the Westinghouse Electric and Manufacturing Co., has been appointed assistant professor of applied mechanics at Harvard University.

Dr. J. L. Dorroh has been appointed research assistant at Princeton University.

Dr. O. J. Farrell has been appointed professor of mathematics at Union College.

Mr. S. E. Field, instructor at the University of Michigan, has been appointed head of the department of mathematics at Junior College, Ironwood, Michigan.

Dr. W. W. Flexner has been promoted to be associate in mathematics at Bryn Mawr College.

Dr. W. O. Menge has been promoted to an assistant professorship in mathematics at the University of Michigan.

Dr. J. E. Powell has been promoted to an assistant professorship of mathematics at Michigan State College.

Assistant Professor M. H. Stone, of Yale University, has been promoted to an associate professorship.

Dr. Oscar Zariski has been promoted to an associate professorship in mathematics at Johns Hopkins University.

Dr. Leo Zippin has been appointed research assistant at Princeton University.

The following appointments to instructorships are announced:

Brooklyn College of the City of New York, Dr. J. M. Feld, Dr. L. S. Kennison, Mrs. Jennie P. Kormes.
University of Minnesota, Mr. Carl H. Fischer.
Northwestern University, Dr. N. E. Rutt, Dr. W. J. Trjitzinsky.
Ohio State University, Dr. L. E. Bush
Pennsylvania State College, Dr. T. C. Benton, Dr. Beatrice L. Hagen.
Princeton University, Mr. M. M. Flood.
Stanford University, Mr. F. A. Butter.
University of Wisconsin, Mr. K. W. Wegner.
Yale University, Mr. W. R. Church.

Dr. L. A. Bauer, Director emeritus of the Department of Terrestial Magnetism of the Carnegie Institute of Washington, died April 12, 1932, at the age of sixty-seven.

Mr. J. D. Grant, of the department of mathematics at the University of Illinois, died July 9, 1932. He had been a member of the Association since 1928.

Professor C. G. Simpson, of the department of mathematics of the School of Engineering, Milwaukee, died February 5, 1932, at the age of sixty.

THE EIGHTH ANNUAL MEETING OF THE LOUISIANA—MISSISSIPPI SECTION

The eighth annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at the University of Mississippi, Oxford, Mississippi, on March 11 and 12, 1932, in connection with the annual meetings of the Louisiana-Mississippi Branch of the National Council of Teachers of Mathematics and the Mississippi Academy of Sciences. Professor A. C. Maddox, Chairman of the Section, presided at the session on Friday afternoon at which there were about thirty in attendance.

At the business meeting the following officers were chosen for next year: Chairman, C. D. Smith, Mississippi State College; vice-Chairmen, T. A. Bickerstaff, University of Mississippi, and H. L. Smith, Louisiana State University; Secretary, May Hickey, Delta State Teachers College. The 1933 meeting of the Section will be held at Ruston, Louisiana.

The Oxford Branch of the American Association of University Women served tea at the Graduate Building on Friday afternoon, and the University of Mississippi entertained at the joint banquet on Friday evening.

Professor Dunham Jackson, University of Minnesota, lectured on "Least Squares and Shortest Distances" at the joint meeting held on Friday evening.

The following papers were presented:

1. "An elementary problem in vector analysis" by Professor T. J. Lindsey, Mississippi State College, by invitation.
2. "Small oscillations of the neutral helium atom near the straight line positions" by Professor H. E. Buchanan, Tulane University.
3. "On Cayley's formulas for orthogonal determinants" by Professor D. S. Dearman, Mississippi State Teachers College.
4. "Trigonometric interpolation" by Professor Dunham Jackson, University of Minnesota.
5. "A study of certain problems from the field of investment" by Professor I. C. Nichols, Louisiana State University.
6. "Some theorems on symmetric determinants" by Professor W. V. Parker, Mississippi Woman's College.
7. "Least squares and shortest distances" by Professor Dunham Jackson, University of Minnesota.

Abstracts of some of these papers follow:

3. The definition of orthogonal determinants is given in terms of $\frac{1}{2}n(n+1)$ independent relations. By use of Cayley's formulas, the n^2 elements of an orthogonal determinant are given in terms of $\frac{1}{2}n(n-1)$ independent parameters. Cayley gives the development of these formulas in Crelle's Journal, Vol. 32. By means of further work done by Brioschi in the 1850's and by Siacci in the